1. Solve the differential equation $y' = \dfrac{3x\sin 2y - 2y}{x - 2x^2\cos 2y}$. (8 分)

2. Solve the differential equation $x^3y'' + 7x^2y' + 9xy = 1 \quad (x > 0)$. (8 分)

3. The Legendre polynomial $P_n(x)$ is a polynomial solution of the Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for $|x| \le 1$, where $n = 0,1,2,\ldots$ and $P_n(1) = 1$.

(1) Show that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ if $m \ne n$. (6 分)

(2) From $(1-2xr+r^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)r^n$ for $|r|<1$,

calculate $\int_{-1}^{1} [P_n(x)]^2\,dx$. (8 分)

4. $J_0(t)$ is a solution of the Bessel equation $t^2y'' + ty' + t^2y = 0$, and $J_0(0) = 1, J_0'(0) = 0$. Calculate the Laplace transform of $J_0(t)$. (10 分)

5. If $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$, find a matrix $\mathbf{Q}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is a diagonal matrix. (10 分)

6. Verify the divergence theorem for the case where the vector field $\mathbf{v} = \hat{\mathbf{j}} + x^2 z\,\hat{\mathbf{k}}$, the volume V: the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. (10%)

7. Verify the Stokes's theorem for the case where the vector field $\mathbf{v} = xz\,\hat{\mathbf{j}}$, and where S is the surface $z = 4 - y^2$, cut off by the planes $x = 0$, $z = 0$, and $y = x$. (10%)

8. Work out the Fourier half-range sine expansion of the given function u(x),

$$u(x) = \begin{cases} 50x, & 0 < x < 1 \\ 100 - 50x, & 1 < x < 2 \end{cases} \qquad (10\%)$$

9. The temperature distribution u(x,t) in a 2-m long brass rod is governed by the problem

$$\alpha^2 u_{xx} = u_t, \quad (0<x<2,\ 0<t<\infty)$$
$$u(0,t) = u(2,t) = 0 \quad (t>0)$$
$$u(x,0) = u(x) \text{ as given in the previous question,}$$

Determine the solution for u(x,t) (10%)

10. Let C is a closed rectangular contour, traversed counterclockwise, with vertices at $-1-i$, $3-i$, $3+3i$, $-1+3i$. Evaluate the given integral.

$$\oint_C \left(\frac{z+1}{z-1}\right)^3 dz \qquad (10\%)$$