國立清華大學命題紙

八十四學年度 化學工程研究 所 甲 組碩士班研究生入學考試

科目 工程數學 科號 1603 共 3 頁第 1 頁 *請在試卷【答案卷】內作答

Problem 1 (20%)

Prove the following gradient operator's relationship when operating on a vector field $\bar{A}$ in three dimensions.

$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

Problem 2 (20%)

Prove the following statements

(a) If A is similar to B, then the characteristic polynomial of B is the same as that of A. 10%

(b) Let A be an n x n matrix. Then A is diagonalizable if and only if there is a set of n linearly 10% independent vectors, each of which is an eigenvector of A.

[Hint: A is similar to B if there exists a nonsingular matrix C such that $B=C^{-1}AC$]

Problem 3 (20%)

(a) Find the Fourier series of the following function: (6%)

$$f(x) = |x|, x \le \pi$$

and show that (6%)

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

(b) Find the inverse Laplace transform of the following function: (8%)

$$\mathcal{L}\{y\} = \frac{s}{(s+2)^2(s^2+2s+10)}$$

Problem 4 (20%)

Consider the following sturm-Liouville equation

$$[p(x)y'(x)]' + [q(x)+\lambda r(x)]y(x)=0 \qquad \text{in } a \le x \le b$$

where p, q, r and p' are real-valued and continuous.

The corresponding boundary conditions are

(1) $k_1 y(a) + k_2 y'(a) = 0$ $\quad (k_1^2 + k_2^2 \neq 0)$

(2) $\ell_1 y(b) + \ell_2 y'(b) = 0$ $\quad (\ell_1^2 + \ell_2^2 \neq 0)$

(A) 8% Prove that the eigenfunctions of the above equation are orthogonal with respect to the weight function $r(x)$.

(B) Find the solution of $y'' + \lambda y = 0$ in $-\pi \leq x \leq \pi$.

(i) $y(\pi) = y(-\pi) = 0$ (4%)

(ii) $y'(\pi) = y'(-\pi) = 0$ (4%)

check if $\lambda = 0$ is an eigenvalue or not.

(C) Define $z = \int \left(\frac{r}{p}\right)^{1/2} dx$, $u(z) = (pr)^{1/4} y$

4% Prove that the Sturm-Liouville equation can be transformed into

$$\frac{d^2u}{dz^2} + [s(z) + \lambda]u = 0$$

find $s(z)$.

Problem 5 (20%)

A semi-infinite slab (see the following figure) was initially with temperature of T_0. At $t=0$, the temperature of the left side to the slab was changed to T_1 and maintained at that temperature afterwards. We wish to determine the time dependent temperature distribution of the slab, $T(x,t)$. Let the thermal diffusivity of the slab be a constant, α, and consider only heat conduction in the x direction.

(a) 3% Give the governing equation, appropriate boundary and initial conditions for $T(x,t)$.

(b) 2% Is the governing equation parabolic, elliptic, or hyperbolic? Is the boundary conditon of Dirichlet, Neumann, or mixed type?

(c) Assume that $T(x,t) = f(q)$ with $q = xt^n$ where n is an unknown constant.

5% Determine n such that the original governing equation (apparently a partial differential equation) reduces to an ordinary differential equation in f and q.

(d) Find $T(x,t)$ through solution of the ordinary differential equation from part (c).
10%

$$\left[\text{Note: } \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-w^2}\,dw\right]$$