國 立 清 華 大 學 命 題 紙

八十四學年度 動力機械研究所 電子、電力 組碩士班研究生入學考試

科目 工程數學 科號 1803 1903 2003 2103 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

## QUESTION 1

(a) Find a series solution in powers of $(x-1)$ of Airy's equation $y'' = xy$, $-\infty < x < \infty$. (10%)

(b) Solve the initial value problem using Laplace transform $y'' + y = f(t)$, $y(0) = 0$ and $y'(0) = 1$

where $f(t) = 1$ if $0 < t < 1$ and $f(t) = 0$ if $t > 1$. (10%)

## QUESTION 2

(a) A, B and C are three vector functions of $t$. If $\frac{d\mathbf{A}}{dt} = \mathbf{C} \times \mathbf{A}$ and $\frac{d\mathbf{B}}{dt} = \mathbf{C} \times \mathbf{B}$, show that

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{C} \times (\mathbf{A} \times \mathbf{B}).$$

(10%)

(b) $S$ is a closed regular surface which bounds the volume $V$, and $\mathbf{n}$ is the unit vector normal to $S$ at a general point. The divergence theorem states

$$\iint_S \mathbf{n} \cdot \mathbf{F}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV.$$

Prove that

$$\iint_S \mathbf{n} \times \mathbf{F}\, dS = \iiint_V \nabla \times \mathbf{F}\, dV.$$

(10%)

## QUESTION 3

For the quadratic form $Q = x^2 + y^2 + z^2 + 2xz + 4\sqrt{2}yz$, it is possible to find an orthogonal matrix $[T]_{3\times 3}$ so that $x, y$ and $z$ are transformed into $u, v$ and $w$, respectively by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [T] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

an.. Q becomes canonical form (i.e. no $uv, uw$ or $wv$ is present). Find [T] and the canonical form of Q.

(20%)

**QUESTION 4**

In presentation of his paper in 1807 before the French Academy of Science, J.B.J. Fourier claimed that any function $f(x)$ whatever defined in the interval $(-\pi, \pi)$, no matter how capriciously, can be represented in that interval by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Unfortunately, Fourier's paper was rejected for lack of rigor.

(a) Derive the Euler formulas for the Fourier coefficients $a_n$ and $b_n$. (12%)

(b) What condition is needed for validity of Fourier's claim ? (4%)

(c) Give an example in which the Fourier coefficients do not exist. (4%)

**QUESTION 5**

Consider the complex function $f(z) = z^5/|z|^4$ with $f(0) = 0$.

(a) Show that $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$. (8%)

(b) Show that $f'(z)$ does not exist at $z = 0$. (8%)

(c) Is $f(z)$ an analytical function at $z = 0$ ? Discuss your point of view. (4%)

[Hint]: For (a), simplify the function by letting $z = x$ on the real axis and letting $z = iy$ on the pure-imaginary axis. For (b), check the value of $f'(z)$ on $z = x$, $z = iy$ and $z = x + ix$ (when $y = x$). Use the notation $f(z) = \phi(x,y) + i\,\psi(x,y)$ and $z = x + iy$ for your answer.