國立清華大學命題紙

八十四學年度 數學 所 ______ 組碩士班研究生入學考試

科目 高等微積分 科號 0101 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (40 pts) Prove or disprove the following statements:
   (a) The set $\{(x, \sin x) : 0 < x \le \pi \text{ and } 0 \le \cos x \le 1/2\}$ is compact.
   (b) Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be continuous. Then $f(\mathbb{R}^2) \ne \mathbb{R} \setminus \{x_0\}$ for each $x_0 \in \mathbb{R}$.
   (c) f is Riemann integrable, where

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational in } [0,1] \\ 0 & \text{if } x \text{ is an irrational in } [0,1]. \end{cases}$$

   (d) Let f be of class C^1 on $\mathbb{R}^2$ and $(a,b) \in \mathbb{R}^2$. Then
   $f(a,b) - f(0,0) = \int_0^1 < \nabla f(ta, tb), (a,b) > dt$.

2. (15 pts) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuous. Prove that for all $T > 0$,

$$\frac{1}{T}\int_0^T \left(\int_{-t}^{t} f(u)\, du\right) dt = \int_{-T}^{T} (1 - \frac{|u|}{T}) f(u)\, du.$$

3. (15 pts) It is known that the following iterated integral can be expressed as a double integral as shown below.

$$\int_0^1 \left(\int_0^{\sqrt{1-y^2}} e^{\sqrt{x^2+y^2}}\, dx\right) dy = \iint_D e^{\sqrt{x^2+y^2}}\, dA.$$

Sketch the region D and evaluate the above double integral.

4. (15 pts) Find the radius of the largest sphere inscribable in the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$.

5. (15 pts) Show that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges uniformly on $[\delta, 2\pi - \delta]$ for $0 < \delta < \pi$.

6. (20 pts) Let $A \subset R^n$ be an open set and $f : A \to R^n$ a C^1 function such that $Jf(x) \ne 0$ for all $x \in A$. Show that f is an open mapping (i.e., $f(\Omega)$ is open in $\mathbb{R}^n$ for each open Ω in A).