國 立 清 華 大 學 命 題 紙

八十四學年度 數 學 所 組碩士班研究生入學考試

科目 代數及線性代數 科號 0102 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (32%)

True or False: Justify your answers.

(a) If G is a group with order $|G| = 2k$, then there is an element $a \neq 1$ such that $a^2 = 1$.

(b) The matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ is diagonalizable over $\mathbb{R}$.

(c) Let A, B be two $n \times n$ matrices over $\mathbb{R}$. If rank $(AB) =$ rank (B), then rank $(A) \geq$ rank (B).

(d) Given two sets of vectors $S_1 = \{x_1, x_2, x_3\}$ and $S_2 = \{y_1, y_2, y_3\}$ in $\mathbb{R}^n$ ($n \geq 3$). Let $V =$ span S_1, $W =$ span S_2. Suppose $\dim_{\mathbb{R}} V = 2 = \dim_{\mathbb{R}} W$. Then there exists a linear map $T : V \to W$, such that $T(x_i) = y_i$, $1 \leq i \leq 3$.

2. (10%)
How many group homomorphisms from the cyclic group $\mathbb{Z}_6$ to the cyclic group $\mathbb{Z}_8$? Prove your answer.

3. (20%)
Define a map $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by $T(A) = A^t$, the transpose of A.
(5%) (a) Show that T is a linear map.
(8%) (b) Determine the eigenvalues of T.
(7%) (c) Determine the dimension of each eigenspace.

4. (15%)
Let T be an orthogonal transformation of $\mathbb{R}^3$, i.e. T is a linear transformation which preserves the inner products in the sense that

$$T(p) \cdot T(q) = p \cdot q,$$

for all $p, q \in \mathbb{R}^3$.
Show that
(a) $\|T(p) - T(q)\| = \|p - q\|$, where $\|\ \|$ is the usual norm on $\mathbb{R}^3$.
(b) Conversely, if F is any transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which preserves the distance and $F(0) = 0$, then F is an orthogonal transformation.

5. (15%)
Prove that, in a commutative ring with identity, every prime ideal of finite index is a maximal ideal.

6. (10%)
Show that the Galois group $\Gamma(\mathbb{R}/\mathbf{Q}) = \{id\}$.
(Hint: If $a < b$ in $\mathbb{R}$, $\varphi \in \Gamma$, then $\varphi(a) < \varphi(b)$).

7. (18%)
Let E be the splitting field over $\mathbf{Q}$ of $x^7 - 1$. How many proper subfields of E between E and $\mathbf{Q}$? Prove your answer.