國立清華大學命題紙

八十四學年度 數學 所 ____ 組碩士班研究生入學考試

科目 複變數函數論 科號 0103 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10 pts)

   (a) Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a connected open set A. If

   $$au(x,y) + bv(x,y) = c \quad \text{in } A,$$

   where a, b, c are real constants not all zero, prove that $f(z)$ is constant in A.

   (b) Is the result valid if a, b, c are complex constants?

2. (5 pts)

   Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. For $r < R$, show that

   $$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

3. (5 pts)

   Suppose f is analytic in Ω, where Ω contains the closed unit disc, and $|f(z)| < 1$ if $|z| = 1$. How many fixed points must f have in the disc?

4. (15 pts)

   Evaluate the following integrals:

   (a) $\int_{|z-\frac{1}{2}|=1} \frac{e^z}{z^3 - z}\, dz$. (b) $\int_0^{\pi/2} \frac{1}{1+\sin^2\theta}\, d\theta$. (c) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx$.

5. (5 pts)

   Find a conformal mapping taking the disc $\{z \in \mathbf{C} : |z-1| < 1\}$ onto the half plane $\{z \in \mathbf{C} : \operatorname{Re} z > 1\}$.

6. (5 pts)

   Suppose f is an entire function and $|f'(z)| \le |z|$ for all z. Show that $f(z) = a + bz^2$ with $|b| \le \frac{1}{2}$.

7. (15 pts)

   (a) Let $u(z)$ be harmonic in $|z-\alpha| < R$. Prove the area mean value theorem:

   $$u(\alpha) = \frac{1}{\pi R^2}\int\int_{|z-\alpha|<R} u(z) r\, dr\, d\theta.$$

   (b) Use the area mean value theorem to prove the maximum principle for harmonic functions.

   (c) Find the maximum of $u = \operatorname{Re}(z^3)$ on the unit square $[0,1] \times [0,1]$.