國 立 清 華 大 學 命 題 紙

八十四學年度 數學 所 組碩士班研究生入學考試

科目 拓樸學 科號 0104 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1.(12 points) Show that $\mathcal{B} = \{(-\infty, a); a \in R\}$ is a basis for a topology $\Im$ on the real line R. Prove or disprove that whether $[0,1]$ in the topology (induced from) $\Im$ is compact? connected? Hausdorff? T_1? (You may use without proof that $[0,1]$ is compact and connected in the usual topology.)

2.(16 points) Let A be a subset of $X = R^n$ carrying the usual metric d. For $x \in R^n$, define $f(x) = d(x, A) = \inf\{d(x,y); y \in A\}$.

(a) Show that f is uniformly continuous.

(b) Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

(c) Let A, B be disjoint closed subsets of X, B compact, show that we can find a small $\epsilon > 0$, such that $U_\epsilon(A) = \{x \in X; d(x, A) < \epsilon\}$ and $U_\epsilon(B)$ are disjoint.

(d) Is the conclusion in (c) still true if B is not compact? Give a proof or a counterexample.

3.(20 points) Regard $G = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in R, ad - bc \neq 0\}$ as a subspace of R^4 in the usual topology.

(a) Show that the map $f : G \times G \to G$ given by $f(g,h) = gh^{-1}$ (product of matrices) is continuous.

(b) For a open subset A, and any subset B of G, show that $AB = \{gh; g \in A, h \in B\}$ is also open in G.

(c) For a compact subset C, and a closed subset D, show that CD is closed.

(d) Is G connected ? Why ?

(e) Let G_o be the connected component of G containing $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $gh^{-1} \in G_o$ whenever $g, h \in G_o$ and $xyx^{-1} \in G_o$ for any $x \in G$ and $y \in G_o$.

4.(12 points) (a) Show that any continuous one to one map $f : [a,b] \to R$ is strictly monotone.

(b) Show that any continuous bijective map $f : R \to R$ is a homeomorphism (in usual topology).