

八十四學年度 應用數學所 組碩士班研究生入學考試

科目 線性代數 科號 0202 共 2 頁第 / 頁 *請在試卷【答案卷】內作答

1. Prove that two similar matrices have the same eigenvalues. (10%)

2. Let A be a real symmetric $n \times n$ matrix. Prove that (15%)

(a) every eigenvalue of A must be real;

(b) two eigenvectors corresponding to two distinct eigenvalues of A must be orthogonal.

3. (15%)

(a) Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(b) Compute

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

4. Let A be a real $m \times n$ matrix and let b be a real m -vector. Prove that $Ax = b$ is solvable if and only if $b^T y = 0$ for all y satisfying $A^T y = 0$. (15%)

5. Given a nonsingular $n \times n$ matrix A . Define a norm

$$\|C\| = \left(\sum_{i,j=1}^n |c_{ij}|^2 \right)^{1/2} \quad \text{if } C = [c_{ij}].$$

Prove that there is a positive constant δ depending on A such that B is nonsingular for any B satisfying $\|B - A\| < \delta$. (15%)

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6. We define a real $k \times k$ matrix A is positive definite if

$$x^T A x > 0 \text{ for all } x \in \mathbb{R}^k \setminus \{0\}.$$

Let B be a real $m \times n$ matrix and let

$$A = \begin{bmatrix} I_m & B \\ B^T & I_n \end{bmatrix},$$

where $I_m(I_n)$ is the $m \times m$ ($n \times n$, respectively) identity matrix. Prove that A is positive definite if and only if $0 \leq \lambda < 1$ for any eigenvalue λ of $B^T B$. (20%)

7. Suppose that A and B are real symmetric positive definite $k \times k$ matrices. Prove that every eigenvalue of AB is positive. (10%)