八十四學年度 應用數學 所 ______ 組碩士班研究生入學考試

科目 微分方程 科號 0203 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

(15%)

1. Let $L[y] = y'' - 2ry' + r^2 y, r \in \mathbb{R}, y = y(t)$

   (a) Show that $L[e^{rt}v(t)] = e^{rt}v''(t)$

   (b) Apply (a) to find a particular solution of

   (*) $$y'' - 6y' + 9y = t^{3/2}e^{3t}$$

   (c) Solve the initial value problem of (*) with initial values $y(0) = 1$, $y'(0) = 0$

2. Let $B = \{x | x : \mathbb{R} \to \mathbb{R}, x(t) \text{ is an } \omega\text{-periodic continuous function}\}$
   If $a \in B$, we define $< a >= \frac{1}{\omega}\int_0^{\omega} a(t)dt$
   Let $a_{11}, a_{12}, a_{22} \in B$ and assume $< a_{22} > \neq 0, < a_{11} > \neq 0$.

   (a) If $(x_1(t), x_2(t))$ is an ω-periodic solution of the system (7%)

   $$\begin{aligned} y_1'(t) &= a_{11}(t)y_1(t) + a_{12}(t)y_2(t) \\ y_2'(t) &= a_{22}(t)y_2(t) \end{aligned}$$

   then $x_1(t) \equiv x_2(t) \equiv 0$ for all t.

   (b) Let $h_1, h_2 \in B$. Show that the system (8%)

   $$\begin{aligned} y_1'(t) &= a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + h_1(t) \\ y_2'(t) &= a_{22}(t)y_2(t) + h_2(t) \end{aligned}$$

   has a unique ω-periodic solution (x_1, x_2).

(15%)

3. Find the series solution $y(x)$ of the following Airy's equation

$$y'' - xy = 0$$

with initial values $y(0) = 0, y'(0) = 1$

(10%)

4. Consider the following second order linear equation

$$\left(p(x)u'\right)' + q(x)u = 0, \quad \text{where } p(x) > 0$$

and $q(x)$ are continuous functions. Introduce the "polar coordinates"

$$\begin{aligned} r &= \left(u^2 + (pu')^2\right)^{1/2} \\ \varphi &= \tan^{-1}\frac{u}{pu'} \end{aligned}$$

Show that

$$\varphi' = \frac{1}{p(x)}\cos^2\varphi + q(x)\sin^2\varphi$$

5. A pond is shaped like a regular cone of radius r and depth d. Water flows into the pond at a constant rate i and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water at time t satisfies the differential equation (8%)

$$\frac{dV}{dt} = i - k\pi\left(\frac{3rV}{\pi d}\right)^{2/3}, \quad \text{for some } k > 0.$$

(b) Find the $\text{limit}_{t\to\infty} V(t)$ without explicitly solving the above equation. (4%)

(c) Control the flow rate i so that the pond will not be overflowed. (3%)

P.S. The volume and the surface area of a regular cone with radius r and height h are $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{h^2 + r^2}$ respectively.