國 立 清 華 大 學 命 題 紙

八十五學年度 數學 系（所）純粹數學組碩士班研究生入學考試

科目 高等微積分 科號 0101 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period of 2π. For $n \in N$, let $f_n(x) = f\left(x + \frac{1}{n}\right)$ for $x \in \mathbb{R}$. Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to f.

2. (15 points) Find the volume of the largest rectangular box that can be inscribed in the ellipsoidal region

$$\left\{(x, y, z) \in \mathbb{R}^3 \middle| \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$$

where $a > 0,\ b > 0,\ c > 0$.

3. (15 points) Find all function $\sigma(x)$ continuous for $x > 0$, and positive real numbers β for which

$$e^x = 2 + \int_\beta^{x^2} \sigma(t)dt.$$

4. Suppose that $f(x)$ is a real, continuously differentiable function on $[a, b]$ with $f(a) = f(b) = 0$ and

$$\int_a^b f^2(x)dx = 10$$

Show that

(a) $\int_a^b xf(x)f'(x)dx = -5$ (7 points)

(b) $\int_a^b (f'(x))^2dx \cdot \int_a^b x^2 f^2(x)dx \ge 25.$ (8 points)

5. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x + y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$. Show that $f(x) = xf(1)$, $\forall x \in \mathbb{R}$.

國立清華大學命題紙

八十五學年度 數學 系（所）純粹數學 組碩士班研究生入學考試
科目 高等微積分 科號 0101 共 2 頁第 2 頁 *請在試卷【答案卷】內作答

6. (15 points) Suppose that $a_n > 0$, $\forall n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges. Set $r_n = \sum_{k=n}^{\infty} a_k$. Show that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

7. Let $S \subsetneq \mathbf{R}^n$ be open and let $f : S \to \mathbf{R}^m$ be differentiable. Suppose that S contains the points a, b and the line segment joining them.

   (a) Show that there exists a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ such that $f(b) - f(a) = L(b - a)$. (8 points)

   (b) Must there be a point c on the line segment joining a and b such that $L = Df(c)$? Justify your answer. (7 points)

8. (15 points) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. Evaluate the integral

$$\int_{S^2} (x^4 + y^4 + z^4) d\sigma,$$

where $d\sigma$ is the surface area element in S^2.