

八十五學年度 數 學 系(所) 純數 組碩士班研究生入學考試
 科目 代數及線性代數 科號 0102 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (12 points)

Let $A \in M_{m \times n}(\mathbb{R})$. Show that if $\text{rank } A = m$, then there exists $B \in M_{n \times m}(\mathbb{R})$ such that $AB = I_m$.

2. (12 points)

Let A, B be $n \times n$ matrices over an infinite field F .

(a) Show that there exists $c \in F$ such that $A + cI_n$ is invertible.

(b) Show that if B is invertible then there exists $c \in F$ such that $A + cB$ is invertible.

3. (20 points)

(a) Let W be a subspace of \mathbb{R}^n (with the standard inner product). Define $W^\perp = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \text{ for all } y \in W\}$. Prove that if $z \in \mathbb{R}^n$, then there exists a unique $y_0 \in W$ such that $z - y_0 \in W^\perp$ and $\|z - y_0\| \leq \|z - y\|$ for all $y \in W$.

(b) Let $A \in M_{m \times n}(\mathbb{R})$, $b \in \mathbb{R}^m$ and let $S = \{x \in \mathbb{R}^n \mid Ax = b\}$. If $S \neq \emptyset$, prove that there exists a unique $x_0 \in S$ such that $\|x_0\| \leq \|x\|$ for all $x \in S$.

4. (16 points)

Let A be a (real) positive definite matrix such that $A^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$.

(a) Compute $\text{tr}(A)$ and $\det(A)$.

(b) Find A .

5. (20 points)

(a) Compute A^n , where $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(b) Let α and β be real numbers and let $\langle a_n \rangle$ be a sequence such that

$$a_1 = \alpha, a_2 = \beta, a_n = \frac{1}{2}(a_{n-1} + a_{n-2}), n \geq 3.$$

Compute a_n in terms of α , β and n .

6. (12 points)

Give an example of a nontrivial homomorphism φ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ into S_3 and justify your answer.

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7. (12 points)

Suppose G is an abelian group of order 24 and G has exactly three elements of order 2. Determine the isomorphism class of G .

8. (16 points)

Let $F = \{0, 1, a, b\}$ be a field with 4 elements. Write down tables of addition and multiplication of the field F and justify your answers.