八十五學年度 數學 系（所）應用數學 組碩士班研究生入學考試
科目 高等微積分 科號 0201 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period of 2π. For $n \in N$, let $f_n(x) = f\left(x + \frac{1}{n}\right)$ for $x \in \mathbb{R}$. Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to f.

2. (15 points) Find the volume of the largest rectangular box that can be inscribed in the ellipsoidal region

$$\left\{(x, y, z) \in \mathbb{R}^3 \middle| \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

where $a > 0,\ b > 0,\ c > 0$.

3. (15 points) Find all function $\sigma(x)$ continuous for $x > 0$, and positive real numbers β for which

$$e^x = 2 - \int_\beta^{x^2} \sigma(t)dt.$$

4. Suppose that $f(x)$ is a real, continuously differentiable function on $[a, b]$ with $f(a) = f(b) = 0$ and

$$\int_a^b f^2(x)dx = 10$$

Show that

(a) $\int_a^b x f(x) f'(x) dx = -5$ (7 points)

(b) $\int_a^b (f'(x))^2 dx \cdot \int_a^b x^2 f^2(x) dx \geq 25.$ (8 points)

5. (15 points) Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is an irrational number.

6. Suppose that $f(x)$ is a differentiable function defined on $\mathbf{R}$. A point p is called a fixed point of f if $f(p) = p$. Show that

   (a) If $f'(x) \neq 1$ for all $x \in \mathbf{R}$, then $f(x)$ has at most one fixed point. (7 points)

   (b) If $|f'(x)| \leq c < 1$ for all $x \in \mathbf{R}$ and some constant c, then $f(x)$ has exactly one fixed point. (8 points)

7. (15 points) Suppose that the coefficients of the power series $\sum_{n=0}^{\infty} a_n x^n$ are related by the equation

$$a_n + Aa_{n-1} + Ba_{n-2} = 0 \quad (n = 2, 3, \cdots).$$

Show that for any x for which the series converges, its sum is

$$\frac{a_0 + (a_1 + Aa_0)x}{1 + Ax + Bx^2},$$

provided $1 + Ax + Bx^2 \neq 0$.

8. (15 points) Let $S^2 = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$. Evaluate the integral

$$\int_{S^2} (x^4 + y^4 + z^4) d\sigma,$$

where $d\sigma$ is the surface area element in S^2.