國 立 清 華 大 學 命 題 紙

八十五學年度 數學 系（所）應用數學 組碩士班研究生入學考試

科目 線性代數 科號 0202 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Let D be the differentiation operator on the space of polynomials of degree less than or equal to 3.
   (a) (5%) Please compute the matrix of D in the basis $u_1 = 1$, $u_2 = 1 + x$, $u_3 = 1 + x^2$, $u_4 = 1 + x^3$.
   (b) (10%) What is the Jordan form of this matrix?

2. (10%) Let A be a real symmetric matrix. Prove that the following two conditions are equivalent.
   (a) All eigenvalues are negative.
   (b) For all $x \in R^n$, $x \neq 0$, we have $x^t A x < 0$.

3. (a) (5%) Let T be a linear operator on a finite dimensional vector space V over the field F. Please give definition of the minimal polynomial for T.
   (b) (10%) Suppose that A is an $n \times n$ matrix with entries in $\mathbb{Q}$. Hence A can be regarded either as an $n \times n$ matrix over $\mathbb{Q}$ or an matrix over $\mathbb{R}$. Is it possible that we can obtain two different minimal polynomials for A. Explain it!

4. (a) (10%) T is a linear operator on V, where V is a finite dimensional vector space over $\mathbb{R}$. T is invertible if and only if the constant term of the minimal polynomial for T is not 0.
   (b) (5%) If T is singular, there exists an operator $S \neq 0$ such that $ST = TS = 0$.

5. (a) (10%) Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$ which are continuous. Let T be the linear operator on V defined by

$$Tf(x) = \int_0^x f(t)dt$$

   Does T have eigenvalue or not? Show it!
   (b) Let V be the vector space over $\mathbb{C}$ consisting of all complex valued differentiable functions of a real variable t.
      i. (5%) Prove that $e^{\alpha_1 t}, e^{\alpha_2 t}, \cdots, e^{\alpha_m t}$ are eigenvectors of the derivative.
      ii. (5%) If $\alpha_i \neq \alpha_j$, for $i \neq j$, show that $e^{\alpha_1 t}, e^{\alpha_2 t}, \cdots, e^{\alpha_m t}$ are linearly independent.

6. (a) (5%) If E is a projection operator and R is the range of E, the vector β is in R if and only if $E\beta = \beta$.

   (b) (10%) If R and N are two subspaces of V such that $R \oplus N = V$, there is one and only one projection operator which has range R and null space N.

   (c) (10%) If $E \neq 0$, prove that there is a matrix C such that

$$CEC^{-1} = \left(\begin{array}{ccc|c} 1 & 0\cdots & 0 & \\ 0 & 1\cdots & 0 & \\ \vdots & & \vdots & 0 \\ 0 & \cdots\cdots & 1 & \\ \hline & 0 & & 0 \end{array}\right)$$

   where the unit matrix in the top left corner is $r \times r$, r is the rank of E.