八十五學年度 數學 系（所） 應數 組碩士班研究生入學考試
科目 微分方程 科號 0203 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (12%) (a) Show that the two families of circles $x^2+y^2-ky=0$ and $x^2+y^2-mx=0$ are orthogonal families.

   (4%) (b) Sketch some curves of $x^2+y^2-ky=0$ and their orghogonal trajectories $x^2+y^2-mx=0$.

2. (12%) (a) Find the general solution of the Legendre equation of order α

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0,\quad -1<x<1.$$

   (4%) (b) Show that there exists exactly a polynomial solution $P(x)$ of the Legendre equation of order $\alpha=2$

$$(1-x^2)y''-2xy'+6y=0,\quad -1<x<1$$

   satisfying $P(1)=1$. Find $P(x)$.

3. Find the solution of the differential equation

$$\begin{cases} y''+y'+\frac{5}{4}y=g(t)=1-u_\pi(t)=\begin{cases}1, & 0\le t<\pi,\\ 0, & t\ge\pi,\end{cases}\\ y(0)=0,\quad y'(0)=0,\end{cases}$$

   (8%) (a) by Laplace transform, and

   (8%) (b) by method of undetermined coefficients and check the solution obtained is the same as the one obtained in (a).

4. (16%) Find the general solution of the linear system

$$\frac{dx}{dt}=x+y+z$$
$$\frac{dy}{dt}=2x+y-z$$
$$\frac{dz}{dt}=-y+z$$

5. (16%)

   Let the functions y_1 and y_2 be linearly independent solutions of the differential equations

$$y''(x)+p(x)y'(x)+q(x)y(x)=0 \text{ on } (-\infty,\infty),$$

   where $p(x)$ and $q(x)\in C(-\infty,\infty)$. Suppose that there exist two real numbers a and b with $a<b$ such that $y_1(a)=y_1(b)=0$ and $y_1(x)\ne 0$ for $x\in(a,b)$. Show that there exists one and only one zero of y_2 on (a,b).
   (Example. $y''(x)+y(x)=0$ on $(-\infty,\infty)$, $y_1(x)=\cos(x)$ and $y_2=\sin(x)$.)
   (Hint. Consider Wronskian $W(y_1,y_2)$.)