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八十五學年度 數學 系（所） 應數 組碩士班研究生入學考試

科目 機率論 科號 0205 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10%)

Two players A and B are going to play a math of a series of 5 independent, and fair games (that is, each player has the same probability $\frac{1}{2}$ to win for each game). The match will continue until either A or B wins 3 games. Find the exptected length of games of the match.

2. (10%)

Let X be a random variable with $EX = 0$. If we define $X^+ = \max\{0, X\}$, then prove that $EX^+ = \frac{1}{2}E|X|$.

3. (15%)

The probability generating function $\Phi_X(t)$ of a nonnegative integer valued random variable X is defined as $\Phi_X(t) = \sum_{x=0}^{\infty} P_r(X = x)t^x$ for $-1 \le t \le 1$. Let $N, X_1, X_2, \cdots, X_n$ be independent nonnegative integer valued random variables. Set $S_N = X_1 + X_2 + \cdots + X_N$. If $X_1, X_2, \cdots X_n$ have the same probability functon with mean μ, then prove that

(1) $\Phi'_N(1) = EN$ (where $\Phi'(t) = \dfrac{d\Phi(t)}{dt}$)

(2) $\Phi_{S_N}(t) = \Phi_N(\Phi_{X_1}(t))$ (you may use the fact if Y and Z are independent random variables, then $\Phi_{Y+Z}(t) = \Phi_Y(t)\cdot\Phi_Z(t)$.)

(3) $ES_N = \mu \cdot EN$

4. (15%)

Let $Y_1, \cdots, Y_r$ be r random variables with the joint probability density function

$$f(y_1, y_2, \cdots, y_r) = \begin{cases} \frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r e^{-\sum_{i=1}^{r}\left(\frac{y_i}{\theta}\right)} e^{-(n-r)\left(\frac{y_r}{\theta}\right)} & \text{for } y_1 < y_2 < \cdots < y_r \\ 0 & \text{otherwise} \end{cases}$$

If $T = \dfrac{\sum_{i=1}^{r} Y_i + (n-r)Y_r}{r}$, then find

(1) the moment generating function of T.
(2) the mean and variance of T.

5. (10%)

Let X and Y are independent random variables with common uniform distribution over $[0,1]$. Prove that $P_r\{X < t | X < Y\} = P_r\{\min(X, Y) < t\}$ for $t \in R$.

6. (10%)

Let $X_1, X_2, \cdots, X_n$ be independent random variables with common moment generating function $M(t) = (\frac{1}{1-\lambda t})^{\alpha}$ for $t < \frac{1}{\lambda}$ and $\alpha > 0, \lambda > 0$. Set $\overline{X} = \sum_{i=1}^{n} X_i/n$.

Prove that $P_r\{\overline{X} > 2\,\alpha\,\lambda\} \le (\frac{2}{e})^{\alpha n}$.

7. (10%)

Let X_n be random variable having Poisson distribution with parameter n for $n \ge 1$.

(1) Prove that $\frac{X_n}{n} \to 1$ in probability (i.e. $P_r\{|\frac{X_n}{n} - 1| > \varepsilon\} \to 0,\ \forall \varepsilon > 0$)

(2) Find $\{a_n\}$ and $\{b_n\}$ such that

$\frac{X_n - a_n}{\sqrt{b_n}} \to N(0,1)$ in distribution (i.e. $P_r\{\frac{X_n - a_n}{\sqrt{b_n}} \le c\} \to \Phi(c),\ \forall c \in R\}$)