ADVANCED CALCULUS
(97 PURE MATH. MASTER ENTRANCE EXAM)

1. (20分) Prove or disprove the following statments:
   (a) Let $\Omega = \{(x, y) \in \mathbf{R}^2 : |x| + |y| \le 1\}$ and $f : \Omega \mapsto \mathbf{R}$ be continuous. Then the set $\{(x, y) \in \Omega : |f(x, y)| \ge 1/3\}$ is compact in $\mathbf{R}^2$.
   (b) Suppose that both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on $\mathbf{R}^2$. Then f is continuous on $\mathbf{R}^2$.

2. (15分) Let $f : \mathbf{R}^2 \mapsto \mathbf{R}$ be of class C^2 and

$$f(2,1) = 0, \quad \frac{\partial f}{\partial x}(2,1) = 1, \quad \frac{\partial f}{\partial y}(2,1) = -1;$$
$$\frac{\partial^2 f}{\partial^2 x}(2,1) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(2,1) = -1, \quad \frac{\partial^2 f}{\partial^2 y}(2,1) = -2.$$

   Set $u(x, y) = f(x^2 + y^2, xy)$. Find $\frac{\partial^2 u}{\partial y^2}(1,1)$.

3. (20分) Let $f : [0,1] \mapsto \mathbf{R}$ be continuous and

$$a_n = (\log n)^{-2} \int_0^1 t^n f(t)\, dt.$$

   (a) Prove that there exists $M > 0$ such that

$$|a_n| \le \frac{M}{(n+1)(\log n)^2} \qquad \text{for all} \quad n \ge 2.$$

   (b) Does the series $\sum_{n=2}^{\infty} a_n x^n$ converge uniformly on $[-1, 1]$? Verify your answer.

4. (15分) Let $f : \mathbb{R}^2 \mapsto \mathbf{R}$ be of class C^1. If $|\nabla f(x, y)| \le 1$ on $\mathbf{R}^2$, prove that

$$\lim_{x \to +\infty} x^{-p} f(x, x) = 0 \quad \text{for all} \quad p > 1.$$

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八十六學年度 數學系 系(所) 純數 組碩士班研究生入學考試

科目 高等微積分 科號 0101 共 2 頁第 2 頁 *請在試卷【答案卷】內作答

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5. (20分)

(a) Let $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta$, and $z = r\cos\phi$.
Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}$.

(b) Evaluate the triple integral $\iiint_\Omega (x^2+y^2+z^2)^{3/2}\,dxdydz$,
where $\Omega = \{(x,y,z) \in \mathbf{R}^3 : 1/2 \le \sqrt{x^2+y^2+z^2} \le 1\}$.

6. (15分) Let Γ be the boundary of the triangle with vertices at $(0,0), (1,0), (0,1)$, traversed once in a counterclockwise direction. Evaluate
$\int_\Gamma xy^2dx + (x^2+y)dy$ by
(a) direct calculation,
(b) Green's theorem.

7. (15分) Let $L(x,\lambda)$ be a C^∞ function on $\mathbf{R}^2$ with $L(0,0) = 0$ and $\frac{\partial L}{\partial x}(0,0) \neq 0$. Show that there are C^∞ functions $\Psi(\lambda)$ in a neighborhood of 0 and $M(x,\lambda)$ in a neighborhood of $(0,0)$ such that

$$L(x,\lambda) = (x - \Psi(\lambda)) \cdot M(x,\lambda),$$

and $\Psi(0) = 0$, $M(0,0) \neq 0$.