國立清華大學命題紙

八十六學年度 數學系 系（所） 應數 組碩士班研究生入學考試

科目 線性代數 科號 0202 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

Linear Algebra(0202)

1.(14%) Denote a point in R^4 by (x_1, x_2, x_3, x_4), let P be the plane in R^4 determined by $x_1 + x_2 + x_3 = 0$, $x_1 - x_2 + x_4 = 0$.

(a) Find the orthogonal projection A from R^4 onto P.

(b) Find $trace(A^{10})$.

2.(14%) Let A denote the reflection on R^3 with respect to the line $x = y = z$, B denote the rotation of 180° on R^3 with z-axis as its rotation axis. Find AB and give the geometric meaning.

3.(10%) Given real matrices $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Are A, B similar? Prove your answer.

4.(18%) Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be a real matrix with $a, b, c, d > 0$.

(a) Show that eigenvalues of A are real and distinct, denote them by $\lambda_1 < \lambda_2$.

(b) Show that $\lambda_2 > 0$, also show that we can find an eigenvector (v_1, v_2) for A with eigenvalue λ_2 such that $v_1, v_2 > 0$.

(c) Show that the sequence $A, A^2, A^3 ...$ has a limit (regarded as points in R^4, i.e. each entry of the sequence converges) if and only if $-1 < \lambda_1 < \lambda_2 \leq 1$.

5.(14%) Let V be the vector space of all $n \times n$ real matrices, V_1, V_2 the subspaces consisting of symmetric, skew-symmetric matrices respectively. Let φ be the symmetric bilinear form on V defined by $\varphi(A, B) = trace(AB)$.

(a) Prove that $\varphi(A, B) = 0$ for $A \in V_1$, $B \in V_2$.

(b) Prove that $\varphi(A, A) > 0$ for $A \neq 0$ in V_1.

6.(14%) Let V be a finite dimensional inner product space; $A, B : V \to V$ be linear transformations. Prove that

(a) $Ker A \subset Ker B$ implies that $B = PA$ for some linear transformation P on V.

(b) $\|Ax\| = \|Bx\|$ for all $x \in V$ implies that $B = PA$ for some isometry P on V.

7.(16%) For column vectors p, q in R^n, define $A = p \cdot q^t - I$ (that is, $a_{ij} = p_i q_j - \delta_{ij}$, where $p^t = (p_1, ..., p_n)$, $q^t = (q_1, ..., q_n)$ and I is the identity matrix).

(a) Find $det A$.

(b) Determine if A is diagonalizable.