# 國立清華大學命題紙

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科目 代數及線性代數 科號 0102 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (10 points) If $A$ is an $n \times n$ matrix such that $A^2 = A$, show that $\mathrm{tr}(A) = \mathrm{rank}(A)$.

2. (20 points) On $\mathbf{R}^4$, let $V$ be the subspace defined by $x_1 = x_4$, $x_2 = x_3$. Denote by $A : \mathbf{R}^4 \to \mathbf{R}^4$ to be the reflection with respect to the subspace $V$.

   (a) Find the matrix (with respect to the standard basis in $\mathbf{R}^4$) representing $A$.

   (b) Find the minimal polynomial of $A$.

3. (20 points) Let $T : V \to V$ be a linear operator. Suppose that $v_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1$ and $v_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2$, where $\lambda_1 \neq \lambda_2$. Put $v = v_1 + v_2$.

   (a) Let $W$ be the $T$-cyclic subspace generated by $v$,(i.e. $W = \mathrm{span}(\{v, T(v), T^2(v), \cdots\})$). Find the dimension of $W$.

   (b) Let $T_W$ be the restriction of $T$ to $W$. Find the characteristic polynomial of $T_W$.

4. (15 points) Let $A$ and $B$ be $m \times n$ matrics. Suppose that $\mathrm{rank}(A) = r_1 \geq \mathrm{rank}(B) = r_2$.

   (a) Prove that $r_1 \leq \mathrm{rank}[A\ B] = \mathrm{rank}[B\ A] \leq r_1 + r_2$ and prove that $\mathrm{rank}[A + B\ B] = \mathrm{rank}[A\ B]$. Where $[A\ B]$ means the $m \times 2n$ matrix, where $A$ is the first sbumatrix, and $B$ is the second matrix, that is $[A\ B] = (c_{ij})$, $c_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq j \leq n \\ b_{i,j-n} & \text{if } n+1 \leq j \leq 2n. \end{cases}$

   (b) Using the results in (a), prove that $r_1 - r_2 \leq \mathrm{rank}(A + B) \leq r_1 + r_2$.

5. (20 points) Let $G$ be a group with identity $e$ and let $N_1$ and $N_2$ be normal subgroups of $G$. Denote $\phi_i : G \to G/N_i$ the canonical epimorphism $\phi_i(a) = aN_i$. Denote $\phi : G \to G/N_1 \times G/N_2$ the unique homomorphism such that $\pi_i\phi = \phi_i$ ($\pi_i$ is the canonical projection of the direct product).

   (a) *Prove that $\phi$ is an isomorphism if and only if $G = N_1 \times N_2$ as internal* direct product (i.e. $G = N_1N_2$ and $N_1 \cap N_2 = \langle e \rangle$).

   (b) Is it true that $\phi$ is an isomorphism if and only if $G \cong N_1 \times N_2$?

6. (15 points) Let $F_2$ be a finite field of 2 elements and let $f(x) = x^4 + x + 1 \in F_2[x]$. Let $\overline{F_2}$ be an algebraic closure of $F_2$.

   (a) Prove that $f(x)$ is an irreducible polynomial over $F_2$.

   (b) Suppose that $\alpha \in \overline{F_2}$ is a root of $f(x)$. Prove that $F_2(\alpha)$ is the splitting field over $F_2$ of $f(x)$, by expressing all other roots of $f(x)$ as elements in $F_2(\alpha)$.

7. (20 points) Let $k$ be a field and $f(x)$ be a polynomial of degree $n$ in $k[x]$. Let $R$ be the residue class ring $k[x]/(f(x))$. In the following, for any $g(x) \in k[x]$, we denote $\overline{g(x)} \in R$ the residue class of $g(x)$ modulo $(f(x))$.

   (a) For any ideal $I \subset R$, prove that there exists $g(x) \in k[x]$ such that $(\overline{g(x)}) = I$ and $g(x)|f(x)$ in $k[x]$.

   (b) Using the result in (a), prove that the total number of distinct ideals of $R$ is not greater than $2^n - 1$. Furthermore, prove that $f(x)$ is irreducible in $k[x]$ if and only if there is no non-zero ideal in $R$.

   (c) Let $I$ be an ideal of $R$. Suppose that $I = (\overline{g(x)})$ and $g(x)|f(x)$ in $k[x]$. Suppose further that the degree of $g(x)$ is $m$. Consider $I$ as a vector space over $k$. Prove that the dimension of $I$ over $k$ is $n - m$ and find a basis for $I$.

   (d) Let $I$ be an ideal of $R$. Suppose that $I = (\overline{g(x)})$ and $g(x)h(x) = f(x)$ for some $h(x) \in k[x]$. Prove that $\overline{\lambda(x)} \in I$ if and only if $f(x)|\lambda(x)h(x)$ in $k[x]$.