國　立　清　華　大　學　命　題　紙

八十七學年度 數學系 系（所） 純粹數學組碩士班研究生入學考試

科目 複變數函數論 科號 0103 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. (14 points) Evaluate the following integrals

   (a)
   $$\int_0^\infty \frac{\sin x}{x(1+x^2)}dx,$$

   (b)
   $$\int_{|z|=7} \tan z dz$$

2. (7 points) Suppose f is holomorphic in $|z| \le 1$, $|f(z)| \le 1$ for $|z| = 1$, $Re\ z \ge 0$ and $|f(z)| \le \sqrt{5}$ for $|z| = 1$, $Re\ z \le 0$. Show that $|f(\frac{1}{2})| \le 2$.

3. (7 points) Suppose $\{f_n\}_{n=1}^{\infty}$ is sequence of harmonic functions on the unit disc U which converges uniformly on every compact subset of U to f, show that f is also harmonic.

4. (8 points) Show that the series $\sum_{n=1}^{\infty} \left(\frac{z-i}{z+i}\right)^n$ defines a holomorphic function on the disc centered at i with radius 1.

5. (8 points) Suppose that $f(z)$ is an entire function such that $|f(z)| \ge |z|^N$ for all $|z| > R$ where $0 < R < \infty$. Show that $f(z)$ must be a polynomial of degree at least N.

6. (7 points) Suppose $f(z)$ is a holomorphic function on $|z| \le 1$ and $|f(z)| < 1$ for $|z| = 1$. Show that there exists unique solution $f(z) = z$ for $|z| < 1$.

7. Let A be the set of all holomorphic functions on $|z| \le 1$ such that $|f(z)| = 1$ for $|z| = 1$.

   (a) (2 points) Find $f_0 \in A$ such that $f_0(\frac{1}{2}) = 0$.

   (b) (7 points) Determine the general form of $f \in A$ and verify your answer.