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八十七學年度 數學系 系(所)應用數學組碩士班研究生入學考試

科目 高等微積分 科號 0201 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1(15 pts). (a) Evaluate the double integral

$$\int_0^1 \int_y^1 \cos(x^2)\, dx dy.$$

(b) Define

$$g(x) = \begin{cases} 3 & \text{if } x \le 0 \\ 3-4x & \text{if } 0 < x < 1 \\ -1 & \text{if } x \ge 1 \end{cases} \quad \text{and} \quad \alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}.$$

Evaluate the Riemann-Stieltjes integral

$$\int_{-3}^3 g(x)\, d\alpha(x).$$

2(10pts). Evaluate the surface integral

$$\int\int_{\sum} (x^4 + y^4)\, d\sigma$$

where $\sum$ is the unit sphere in $\mathbf{R}^3$ and $d\sigma$ is the surface element on $\sum$.

3(20pts). Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} \sqrt{x^2+y^2}\, \sin\left(\frac{x^2}{y}\right) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

(a) Show that f is continuous at the point $(0,0)$.
(b) Show that f has directional derivatives in every direction at $(0,0)$.
(c) Is f differentiable at $(0,0)$? Explain.

4(15 pts). (a) Show that the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k\sqrt{k}}$ converges uniformly on $\mathbf{R}$.
(b) Does there exist a polynomial $p(x)$ such that

$$\left| p(x) - \sum_{k=1}^{\infty} \frac{\sin kx}{k\sqrt{k}} \right| < 10^{-3} \quad \text{for all } x \in (0,1) \text{ ?}$$

Show your reason.

5(10 pts). Let $f:\mathbf{R}\to\mathbf{R}$ be a bounded function. Does there exist a sequence of positive integers $n_1 < n_2 < n_3 < \cdots \to \infty$ such that $\lim_{k\to\infty} f(n_k)$ exists? Show your reason.

6(15 pts). Let $S=\{(x,y): x^2+y^2=1\}$ be the unit circle in $\mathbf{R}^2$, and let $f: S\to\mathbf{R}$ be a continuous function. Prove that there are two antipodal points (x_0,y_0) and $(-x_0,-y_0)$ in S such that $f(x_0,y_0)=f(-x_0,-y_0)$.

7(20 pts). (a) Let $h:\mathbf{R}\to\mathbf{R}$ be a differentiable function, and suppose that there is a constant $c>0$ such that $h'(t)\geq c$ for all $t\in\mathbf{R}$. Prove that there is exactly one point t at which $h(t)=0$.
(b) Let $f:\mathbf{R}^2\to\mathbf{R}$ be a C^1 function, and suppose that there is a constant $c>0$ such that

$$\frac{\partial f}{\partial y}(x,y)\geq c \quad \text{for all } (x,y)\in\mathbf{R}^2.$$

Prove that there is a C^1 function $g:\mathbf{R}\to\mathbf{R}$ with $f(x,g(x))=0$ for all $x\in\mathbf{R}$.

8(15 pts). Let $f(x,y)$ be a continuous real-valued function defined on the closed unit disc $\overline{\Delta}$, where $\Delta=\{(x,y): x^2+y^2<1\}$. Suppose that f satisfies the submean value property on Δ, i.e., for any $p=(x_0,y_0)\in\Delta$ and any $0<r<1-\sqrt{x_0^2+y_0^2}$, we have

$$f(p)\leq\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta.$$

Prove that if f is not a constant function, then f must achieves its maximum on the boundary of Δ.