1. (10 points) If A is an $n \times n$ matrix such that $A^2 = A$, show that $\operatorname{tr}(A) = \operatorname{rank}(A)$.

2. (15 points) On $\mathbf{R}^4$, let V be the subspace defined by $x_1 = x_4$, $x_2 = x_3$. Denote by $A : \mathbf{R}^4 \to \mathbf{R}^4$ to be the reflection with respect to the subspace V.

   (a) Find the matrix (with respect to the standard basis in $\mathbf{R}^4$) representing A.

   (b) Find the minimal polynomial of A.

3. (15 points) If $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, and if we know that $A^{10} = \alpha A + \beta I$, find α, β.

4. (10 points) If p, q are two orthonormal column vectors in $\mathbf{R}^n$, and $A = p \cdot p^T + q \cdot q^T$, where p^T is the transpose of p. Find the characteristic polynomial of A.

5. (10 points) If A, B are two symmetric matrices. Denote by $\lambda_1(A)$ the smallest eigenvalue of A. Show that

$$\lambda_1(A + B) \geq \lambda_1(A) + \lambda_1(B).$$

6. (15 points) If $A = (a_{ij})_{1 \leq i, j \leq n}$ where $a_{ij} = 1$.

   (a) Find the rank of $A - xI$ in terms of x.

   (b) Find the optimal x so that $A - xI$ is positive definite.

7. (15 points) Let $T : V \to V$ be a linear operator. Suppose that v_1 is an eigenvector corresponding to the eigenvalue λ_1 and v_2 is an eigenvector corresponding to the eigenvalue λ_2, where $\lambda_1 \neq \lambda_2$. Put $v = v_1 + v_2$.

   (a) Let W be the T-cyclic subspace generated by v, i.e. $W = \operatorname{Span}\{v, Tv, T^2 v \cdots\}$. Find the dimension of W.

   (b) Let T_W be the restriction of T to W. Find the characteristic polynomial of T_W.

國立清華大學命題紙

八十七學年度 數學系 系(所) 應用數學 組碩士班研究生入學考試

科目 線性代數 科號 0202 共 2 頁第 2 頁 *請在試卷【答案卷】內作答

8. (10 points) Let A and B be $m \times n$ matrics. Suppose that $\text{rank}(A) = r_1 \geq \text{rank}(B) = r_2$.

   (a) Prove that $r_1 \leq \text{rank}[A\ B] = \text{rank}[B\ A] \leq r_1 + r_2$ and prove that $\text{rank}[A + B\ B] = \text{rank}[A\ B]$. Where $[A\ B]$ means the $m \times 2n$ matrix, where A is the first sbumatrix, and B is the second matrix, that is $[A\ B] = (c_{ij})$, $c_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq j \leq n \\ b_{i,j-n} & \text{if } n+1 \leq j \leq 2n. \end{cases}$

   (b) Using the results in (a), prove that $r_1 - r_2 \leq \text{rank}(A + B) \leq r_1 + r_2$.