1. (10 points) Show that $|\sin^2 x - \sin^2 y| \le |x - y|$ for any $x, y \in \mathbb{R}$.

2. (10 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function, and let S be a surface defined by $S = \{(x,y,z) : f(x,y,z) = c\}$ for some constant c. Prove that $\nabla f(x,y,z)$ is perpendicular to S if $(x,y,z) \in S$.

3. (16 points) Suppose $f : (-5,5) \to \mathbb{R}$ is a continuous function. Which of the following statements must be true and which could be false? (Give reasons for your answers)

   a) The set $\{f(x) : 0 < x < 1\}$ is open.

   b) The set $\{f(x) : 0 < x < 1\}$ is bounded.

   c) f is uniformly continuous on the interval $(-1,1)$.

4. (12 points) Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Prove that there exists a point $c \in [a,b]$ such that $\int_a^c f(x)\,dx = \int_c^b f(x)\,dx$.

5. (12 points) Show that the continuous function $f(x,y) = (xy)^{1/3}$ has partial derivatives $f_x(0,0)$ and $f_y(0,0)$, but f is not differentiable at $(0,0)$.

6. (12 points) Find

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$$

by considering Riemann sums for $\int_0^1 \log x\,dx$ based on the partition $\{\frac{1}{n}, \frac{2}{n}, \cdots \frac{n}{n}\}$.

7. (12 points) Let ϵ be a positive number. Show that there is no function f satisfying the following conditions: $f'(x)$ exists for $x \ge 0$, $f'(0) = 0$ and $f'(x) \ge \epsilon$ for $x > 0$.

8. (12 points) Let $f(x,y)$ be continuous on $[a,b] \times [c,d]$, and suppose $\{\varphi_n(x)\}$ converges uniformly on $[a,b]$ with $c \le \varphi_n \le d$. Show that the sequence $F_n(x) = f(x, \varphi_n(x))$ converges uniformly on $[a,b]$.

9. (12 points) Evaluate the line integral

$$\oint_{\partial D} \frac{dx + dy}{|x| + |y|},$$

where D is the rectangle with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.

10. (12 points) Consider the family of mappings from $\mathbf{R}^2$ to $\mathbf{R}^2$ defined by

$$F_t(x,y) = (x + ty^2, x + y), \quad t \in \mathbf{R}.$$

Let $C = [0,1] \times [0,1]$, and set $C_t = F_t(C)$.

a) Show that the set $\bigcup_{t \in \mathbf{R}} C_t$ is connected.

b) Is the set $\bigcup_{t \in \mathbf{R}} C_t$ compact? Explain.