1. (20%)

Denote by $P_2(\mathbf{R})$ the set of polynomials of degree ≤ 2. $S, T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ are defined by

$$T\left(ax^2 + bx + c\right) = cx^2 + (a+b)x + a$$
$$S\left(ax^2 + bx + c\right) = cx^2 + bx + a.$$

Test if T, S are diagonalizable or not. If diagonalizable, find a basis β such that the corresponding matrix representation is diagonal. Otherwise give a reason.

2. (10%)

If $a, b \in \mathbf{C}$, $A = \begin{bmatrix} 2 & a \\ b & 3 \end{bmatrix}$, $< x, y >= \bar{x}^t A y$ for $x, y \in \mathbf{C}^2$. Find conditions on a, b so that $< \, , \, >$ is a inner product in $\mathbf{C}^2$.

3. (15%)

Let $u = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, v = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, W = \{x \in \mathbf{R}^4 | \ u^t x = v^t x = 0\}$.

(a) Find the projection P of $\mathbf{R}^4$ onto W.

(b) If $y = \begin{bmatrix} 3 \\ 0 \\ 0 \\ b \end{bmatrix}$. Find b so that $(I - P)x = y$ is solvable, and then find all the solutions x.

4. (15%)

If $A = (a_{ij}), B = (b_{ij})$ are $n \times n$ matrices

$$a_{ij} = \begin{cases} 0 & \text{if } \; i = j \\ 1 & \text{if } \; i \neq j \end{cases}$$

$$b_{ij} = \begin{cases} 2 & \text{if } \; i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the characteristic polynomial of A

(b) Find $det(A+B)$.

5. (15%)

Let R be a commutative ring with 1, and let M be a maximal ideal of R. Then prove that R/M is a field.

6. (15%)

Let R be the ring of all rational numbers having odd denominators in their reduced form. Then prove

$$(2) = \{2a \mid a \in R\} \subseteq R$$

is a maximal ideal of R.

7. (15%)

Let G be the group of nonzero real numbers under multiplication and let $N = \{1, -1\}$. Prove that $G/N \cong$ positive real numbers under multiplication.

8. (15%)

Let G, G' be groups and let the map $f : G \to G'$ be a group homomorphism of G onto G'. If H' is a normal subgroup of G' and if

$$H = \{a \in G | \ f(a) \in H'\},$$

show H is a normal subgroup of G.