1. (15 points, 5 points for each part) Let $U = \{(x_1,x_2,x_3,x_4)^T \mid x_1 + 2x_2 + 3x_3 = 0\}$ and $V = \{(x_1,x_2,x_3,x_4)^T \mid x_1 + x_3 + 2x_4 = 0\}$ be two subspace of $\mathbf{R}^4$.

   (a) Find a basis of $U \cap V$.

   (b) Extend the basis you find in (1a) for $U \cap V$ to a basis for U and to a basis for V.

   (c) Use the results in (1b) to prove that $U + V = \mathbf{R}^4$.

2. (15 points, 5 points for each part) Let $\overrightarrow{e_1} = (1,0,0)^T$, $\overrightarrow{e_2} = (0,1,0)^T$ and $\overrightarrow{e_3} = (0,0,1)^T$ be the standard basis of $\mathbf{R}^3$. For fixed $a,b,c,f,g,h \in \mathbf{R}$, we define a map $L : \mathbf{R}^3 \to \mathbf{R}$ by

$$L(x\overrightarrow{e_1} + y\overrightarrow{e_2} + z\overrightarrow{e_3}) = \det \begin{pmatrix} a & x & f \\ b & y & g \\ c & z & h \end{pmatrix}.$$

   (a) Verify that L is a linear mapping and find the matrix representing L with respect to the ordered basis $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$

   (b) Suppose that $v = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$ and $w = f\overrightarrow{e_1} + g\overrightarrow{e_2} + h\overrightarrow{e_3}$ are linearly independent. Find the dimension of the kernel of L.

   (c) Suppose that $v = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$ and $w = f\overrightarrow{e_1} + g\overrightarrow{e_2} + h\overrightarrow{e_3}$ are linearly dependent. Find the dimension of the kernel of L.

3. (30 points, 5 points for each part) For an $n \times n$ matrix over $\mathbf{R}$, we define the null space $N(A) = \{v \in \mathbf{R}^{n\times 1} = \mathbf{R}^n \mid Av = 0\}$ and we define the range $R(A) = \{Av \mid v \in \mathbf{R}^n\}$.
   For $v = (a_1,a_2,a_3)^T$, $w = (b_1,b_2,b_3)^T \in \mathbf{R}^3$, define $< v, w >= a_1b_1 + a_2b_2 + a_3b_3$ and $\| v \| = \sqrt{< v, v >}$. For a subspace V of $\mathbf{R}^3$, we define $V^\perp = \{w \in \mathbf{R}^3 \mid < w, v >= 0, \forall v \in V\}$.
   Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ -10 \\ 20 \end{pmatrix}.$$

   (a) Find a basis of $R(A)^\perp$.

   (b) Find the unique $p \in R(A)$ which is closest to b.

   (c) Let $S = \{x \in \mathbf{R}^3 \mid Ax = p\}$ be the set of solutions to the system $Ax = p$. Find S.

   (d) Find an orthonormal basis of $N(A)^\perp$.

   (e) Find the unique $v_0 \in S$ such that $v_0 \in N(A)^\perp$.

   (f) Let v_0 be as in (3e). Show that for all $v \in S$ such that $v \neq v_0$, we have $\| v \| > \| v_0 \|$.

4. (15 points, 5 points for each part) For a complex $n \times n$ matrix $A = (a_{ij})_{1\le i,j\le n}$, we define its trace to be $trA = a_{11} + a_{22} + \ldots + a_{nn}$. Consider the set $\mathbb{S}$ of all complex $n \times n$ matrices satisfying $A^m - I = 0$ for a (fixed) positive integer m.

   (a) Prove that $|trA| \le n$ for any $A \in \mathbb{S}$.

   (b) Find the subset $\{A \in \mathbb{S}; |trA| = n\}$.

(c) Find the subset $\{A \in \mathbb{S}; tr A = n\}$.

5. (10 points, 5 points for each part) Given an idempotent matrix A, i.e. one which satisfying $A^2 = A$.

   (a) Show that $B = I - 2A$ is involutive, i.e. $B^2 = I$. Therefore B is nonsingular.

   (b) Find all scalars λ for which $I - \lambda A$ is nonsingular.

6. (15 points, 5 points for each part) Let S be a nonempty finite set, for any given $s \in S$, denote by χ_s the real valued function on S given by $\chi_s(t) = 0$ for any $t \neq s$ and $\chi_s(s) = 1$. Consider the vector space $V = \{f : S \to \mathbf{R}\}$ of all real-valued functions on S, and let $\varphi : S \to S$ be a map.

   (a) Show that $\Phi : V \to V$ defined by $\Phi(f)(t) = f(\varphi(t))$ $(t \in S)$ is linear.

   (b) Show that $\{\chi_s; s \in S\}$ is a basis for V.

   (c) Show that the trace of Φ equals to the number of fixed points of φ (a point $t \in S$ is called a fixed point of φ if $\varphi(t) = t$).