九十學年度 數 學 系（所） 純 數 組碩士班研究生招生考試

代數及線性代數 科號 0102 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

20% 1.

(a) Prove that $Q(\sqrt{3}+\sqrt{7}) = Q(\sqrt{3},\sqrt{7})$, both are the extension fields of the rational numbers.

(b) Prove that x^2-3 is irreducible over $Q(\sqrt[3]{2})$.

10% 2. If p is a prime, show that

$$f(x) = x^{p-1} + x^{p-2} + x^{p-3} + \cdots + x + 1$$

is irreducible in $Q[x]$.

15% 3. Let p be a fixed odd prime, and let

$$R = \{\frac{b}{a} \mid (a,b)=1,\ (a,p)=1,\ a,b \text{ are integers }\}$$

be a subring of the rational numbers. Then prove

$$(p) = \{pr \mid r \in R\} \subseteq R$$

is a maximal ideal of R.

15% 4. If G is a group of order p^n ($n \geq 1$), where p is a prime, then prove that the center of G is not trivial (that is, there is an element $a \neq e$ in G such that $ax = xa$ for all $x \in G$, where e the unit element of G).

20% 5. Let $f: V \to W$ be a linear mapping from the finite dimensional vector space V to the finite dimensional vector space W such that f is surjective, then prove that $V \cong Ker f \oplus W$, where $Ker f$ is the null space of the mapping f and $\oplus$ means the direct sum.

15% 6. Find the minimal polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

and from the minimal polynomial what can you say about the characteristic polynomial ?

10% 7. Let V be the vector space of all 2×2 matrices over the real numbers and let P be a fixed 2×2 matrix. Let $T: V \to V$ be a linear mapping defined by $T(A) = PA$. Prove that $trace(T) = 2\ trace(P)$.

15% 8. A, B are real $n \times n$ matrices, and A, AB are symmetric with A positive definite. Show that B has real eigenvalues.