NTHU MSc Program Entrance Exam, April 14, 2001

Linear Algebra (Applied Math Program)

Total exam time: 100 minutes. Total 100 points.

(1) (8 %) Let l_∞ be the vector space of bounded sequences with usual addition and scalar multiplication. Which, if any, of the following sets are bases for l_∞? Explain.

- (a) $E = \{(1,0,0,\cdots),(0,1,0,0,\cdots),\cdots(0,\cdots,0,1,0,\cdots),\cdots\}$
- (b) $F = \{(1,1,1,\cdots),(0,1,1,1,\cdots),(0,0,1,1,1,\cdots),\cdots\}$

(2) (12%) Denote by $\mathcal{M}_{2,2}$ the vector space of all 2×2 matrices and $P \in \mathcal{M}_{2,2}$ a fixed matrix. Let $T : \mathcal{M}_{2,2} \longmapsto \mathcal{M}_{2,2}$ be the linear mapping defined by $T(A) = PA$. Prove that $\mathrm{trace}(T) = 2\,\mathrm{trace}(P)$.

(3) Let $A,B \in \mathcal{M}_{n,n}$. Show that

- (a) (15%) row rank of $AB \leq$ row rank of B.
- (b) (5 %) Use (a) to show that (row rank of AB = row rank of B) if A is nonsingular.

(4) (15%) Let $A \in \mathcal{M}_{m,m}$, $B \in \mathcal{M}_{m,n}$ and $C \in \mathcal{M}_{n,n}$. Show that

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = (\det A)(\det C)$$

(5) (15%) Let A be a real $n\times n$ matrix with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Denote the corresponding left and right eigenvectors by l_i and r_i, $i = 1,\cdots,n$ respectively. Show that $\sum_{i=1}^{n} r_i l_i$ is a nonsingular matrix.

(6) (15%) Classify all 9×9 matrices with minimal polynomial $(x-2)^3(x-4)^2$ according to their Jordan canonical forms. Show your work.

(7) (15%) Let A, B be real $n\times n$ matrices with A and AB symmetric. In addition, A is positively definite. Show that the eigenvalues of B are real.