國 立 清 華 大 學 命 題 紙

九十三學年度 科技管理研究所 系（所） 乙 組碩士班入學考試

科目 微積分 科號 6003 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

I. 填充題(共六題,每題9分,請將答案依甲,乙,丙,...,次序作答,不需演算過程)

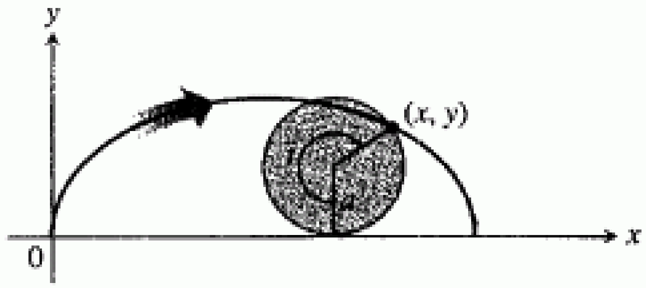
1. Suppose $\lim_{x\to 1}\dfrac{\sqrt{ax+b}-2}{x-1}=1$. Then $a-b=$ ___甲___.

2. Let $d\in\mathbf{R}$. Suppose $1+e^{d}+e^{2d}+e^{3d}+\cdots+e^{nd}+\cdots=9$. Then $d=$ ___乙___.

3. Compute $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{dx}{1+\cos x}=$ ___丙___.

4. Let A be the **area of the largest triangle** that is symmetric (對稱) about the x-axis and can be put inside the ellipse $x^2+4y^2=1$. Then $A=$ ___丁___.

5. Let R be the **area of the region** bounded below by the x-axis and above by one arch of the cycloid (輪轉線) parameterized by $x=a(t-\sin t)$, $y=a(1-\cos t)$, $0\le t\le 2\pi$, see next figure. Then $R=$ ___戊___.



6. Let L be the **line integral** of $f(x,y,z)=x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by

$$C_1: r=(t,t^2,0),\ 0\le t\le 1,$$
$$C_2: r=(1,1,t),\ 0\le t\le 1.$$

Then $L=\int_{C_1\cup C_2}f(x,y,z)ds=$ ___己___. (Note. $ds=\sqrt{(dx)^2+(dy)^2+(dz)^2}$)

II.計算與証明(必須寫出演算証明過程)

1. (5%) (a) State the **integral test**.

   (7%) (b) Show that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ converges if $p>1$ and diverges if $0<p\le 1$ by the **integral test**. (Note. Check the conditions.)

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2. **Taylor's remainder theorem.** The Taylor series of a function f around zero is given by

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + R_N(x),$$

where $f^{(n)}$ is the n-th derivative of f, and the remainder term R_N is given by

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

for some point c between 0 and x. (Note. You do *not* need to prove Taylor's remainder theorem.)

Problem.

(a) (2%) Write this series for the function e^x for a general N.

(b) (3%) Show that for e^x,

$$|R_N(1)| \le \frac{e}{(N+1)!}.$$

(c) (5%) Apply Taylor's remainder theorem and parts (1) and (b) to show that

$$\frac{15}{7} < e < 3.$$

3. Let $f(x,y) = (y - x^2)(y - 2x^2)$.

(a) (6%) Show that f has neither a local minimum nor a local maximum at $(0,0)$.

(b) (6%) Show that f has a local minimum at $(0,0)$ when considered on any fixed line through $(0,0)$.

4. (12%) The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Apply the method of Lagrange multiplier to find the point(s) on the ellipse that lie(s) farthest (最遠) from the origin $(0,0,0)$, see next figure.