國立清華大學命題紙

九十一學年度 數學 系（所） 純粹數學組 碩士班研究生招生考試

科目 高等積幾分 科號 0101 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15 points) Evaluate the integral

$$\int_0^\infty e^{-7x^2}\,dx.$$

2. (15 points) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function. Suppose that

$$x \cdot \nabla f(x) = kf(x).$$

Prove that

$$f(\lambda x) = \lambda^k f(x) \text{ for each } x \in \mathbf{R}^n \text{ and } \lambda > 0.$$

3. (15 points) How many real numbers x satisfy

$$x + 1 = 4\arctan(x)?$$

4. (15 points) Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[a, b]$ such that $f_n(a) = f_n(b) = 0$ and

$$\int_a^b |f_n'(x)|^2 dx \le M$$

for some fixed constant M. Prove that $\{f_n\}$ contains a subsequence that converges uniformly on $[a, b]$.

5. (15 points)

   (a) Let f be a continuous function from $[0, 1]$ onto $[0, 1]$, show that there exists a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

   (b) If g is a continuous function from $(0, 1)$ onto $(0, 1)$, can there always exists a fixed point in $(0, 1)$ for g? Justify your answer.

6. (15 points) Define

$$f(x,y) = \begin{cases} \frac{x^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

   (a) Show that f is continuous at $(0,0)$.

   (b) Show that f has directional derivatives in all directions at $(0,0)$.

   (c) Show that f is not differentiable at $(0,0)$.

7. (15 points) Suppose f is defined in $S = [0,1] \times [0,1]$ by the formula

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is irrational}, \\ 3y^2 & \text{if } x \text{ is rational}. \end{cases}$$

   (a) Show that $\int_0^1 (\int_0^1 f(x,y)dy)dx$ exists and has the value 1.

   (b) Show that $\int \int_S f(x,y)dA$ does not exist.

8. (15 points) Check whether the following sets E in $\mathbf{R}^2$ are compact or not. If E is not compact, find the smallest compact set K such that $E \subset K$. Justify your answers!

   (a) $E = \{(x,y) : y = \sin(\frac{1}{x}),\, x \in (0,1]\}$.

   (b) $E = \{(x,y) : x \in [0,1]$ and $y = 1$ if x is rational, $y = 0$ if x is irrational$\}$.