九十一學年度 數 學 系（所） 純粹數學組 碩士班研究生招生考試

科目 代數及線性代數 科號 0102 共 2 頁 第 1 頁 *請在試卷【答案卷】內作答

Algebra and Linear Algebra (總分 120 分)

(12%) 1.

Determine "true" or "false" for the following statements (without proofs).

(a) $det: M_n(\mathbf{R}) \longrightarrow \mathbf{R}$ is linear over $\mathbf{R}$, where $M_n(\mathbf{R})$ is the vector space of all the $n \times n$ matrices over $\mathbf{R}$.

(b) If $A, B \in M_n(\mathbf{R})$ are similar then they have the same eigenvectors.

(c) For $A, B \in M_n(\mathbf{R})$ if $AB = I_n$ (identity matrix) then $BA = I_n$.

(d) $rank(AB) \geq rank B$ for any $A, B \in M_n(\mathbf{R})$.

(16%) 2.

Let V be an n-dimensional vector space over $\mathbf{R}$ and $V \xrightarrow{T} V$ be a linear transformation such that the range and null space of T are identical.

(a) Prove that n must be even. (8%)

(b) Give an example of such a linear transformation for $V = \mathbf{R}^2$. (8%)

(15%) 3.

Show that any $A \in M_n(\mathbf{R})$ which is upper triangular and orthogonal (means $AA^T = I_n$) is a diagonal matrix.

(17%) 4.

(a) For $A \in M_n(\mathbf{R})$, if $Av = 0$ for some $v \neq 0$ in $\mathbf{R}^n$ prove that $det A = 0$. (7%)

(b) Let $f(x)$ be the characteristic polynomial of a matrix $B \in M_n(\mathbf{R})$. If 1 is an eigenvalue of B, prove that $det f(B^2) = 0$. (10%)

(15%) 5.

Prove that any group of order 78 has a normal subgroup of order 39.

(12%) 6.

Determine whether the two abelian groups in each of the following pairs are isomorphic to each other and explain why ?

(a) $(\mathbb{Q}/\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$.

(b) $(\mathbb{R}^*, \cdot)$ and $(\mathbb{R}^+, \cdot)$ where $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

(c) $\mathbb{Z}_6 \times \mathbb{Z}_{15} / < (2,3) >$ and $\mathbb{Z}_6$.

(18%) 7.

Consider the Gaussian ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Let $< 2 + 3i >$ be the principal ideal of $\mathbb{Z}[i]$ generated by $2 + 3i$. Note that $N(2 + 3i) = (2 + 3i)(2 - 3i) = 4 + 9 = 13$.

(a) Prove that the inclusion map $\mathbb{Z} \xrightarrow{j} \mathbb{Z}[i]$, $a \xrightarrow{j} a = a + 0i$, which is a ring homomorphism, induces a ring homomorphism $\mathbb{Z}/13\mathbb{Z} \xrightarrow{\bar{j}} \mathbb{Z}[i]/ < 2 + 3i >$. (5%)

(b) Prove that $\bar{j}$ is 1-1. (6%)

(c) Prove that $\bar{j}$ is onto. (7%)

(15%) 8.

Let p be an odd prime. The polynomial $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is well known to be irreducible over $\mathbb{Q}$ and $\zeta = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$, $\zeta^2, \cdots, \zeta^{p-1}$ are zeros of $\Phi_p(x)$ in $\mathbb{C}$. Consider the extension field $\mathbb{Q}(\zeta) \subset \mathbb{C}$.

(a) Show that the Galois group $G(\mathbb{Q}(\zeta)/\mathbb{Q})$ is an abelian group of order $p - 1$. (7%)

(b) Show that $\left|G(\mathbb{Q}(\zeta + \frac{1}{\zeta})/\mathbb{Q})\right| = \frac{p-1}{2}$. (8%)