Linear Algebra (總分 100 分)

(16%) 1.

Determine "true" or "false" for the following statements and give proofs for the true ones.

(a) $det: M_n(\mathbf{R}) \longrightarrow \mathbf{R}$ is linear over $\mathbf{R}$, where $M_n(\mathbf{R})$ is the vector space of all the $n \times n$ matrices over $\mathbf{R}$.

(b) If $A, B \in M_n(\mathbf{R})$ are similar then they have the same eigenvectors.

(c) For $A, B \in M_n(\mathbf{R})$ if $AB = I_n$ (identity matrix) then $BA = I_n$.

(d) $A \in M_4(\mathbf{R}) \Longrightarrow \det(-A) = -\det A$.

(14%) 2.

Let H be the linear subspace of $\mathbf{R}^4$ spanned by the vectors $(1,1,1,1), (1,0,1,1)$ and $(0,1,1,1)$. Find the orthogonal projection of the vector $(2,3,3,1)$ on H.

(14%) 3.

Let V be an n-dimensional vector space over $\mathbf{R}$ and let $V \xrightarrow{T} V$ be a linear transformation such that the range and null space of T are identical.

(a) Prove that n must be even.

(b) Give an example of such a linear transformation for $V = \mathbf{R}^2$.

(13%) 4.

Let A be an $m \times n$ matrix over $\mathbf{R}$. Suppose for every $b \in \mathbf{R}^m$, $Ax = b$ has at least one solution x in $\mathbf{R}^n$. Prove that $A^T y = 0$ has only one solution in $\mathbf{R}^m$ where A^T is the transpase of A.

(13%) 5.

Show that any $A \in M_n(\mathbf{R})$ which is upper triangular and orthogonal (means $AA^T = I_n$) is a diagonal matrix.

(16%) 6.

Determine, up to similarity, all $A \in M_3(\mathbf{R})$ with $A^3 = A$.

(14%) 7.

Let V be an n-dimensional vector space over $\mathbf{R}$. Let $(v,w) \longrightarrow < v,w >$ be a non-singular bilinear form on $V \times V$. Let $c \in \mathbf{R}$, and let $V \xrightarrow{A} V$, $V \xrightarrow{B} V$ be linear transformations such that $< Av, Bw > = c < v,w >$ for all $v, w \in V$. Prove that $\det A \cdot \det B = c^n$.