國 立 清 華 大 學 命 題 紙

九十二學年度 數 學 系（所） 應用數學 組碩士班研究生招生考試

科目 高等微積分 科號 0201 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

***Show your work, otherwise no credit will be granted.**
****Each problem is worth 15 points.**

1. Let S be the surface: $x^2+y^2+z^2=1$, $z \geq 0$ with unit normal $\overrightarrow{n}$ pointing out(up)ward. Suppose that $\overrightarrow{F} = y^3\mathrm{i} + 2yz\mathrm{j} + (x^2+y^2-z^2)\mathrm{k}$. Evaluate

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, d\sigma,$$

where $d\sigma$ is the surface element.

2. Let D be a bounded closed region in $\mathbb{R}^3$ enclosed by the surfaces: $z=2y$, $z=0$ and $y=2-x^2$. Suppose that $f(x,y,z)=x+2y+3z$, find the extrema of f on D.

3. Let $f_n(x)=\cos(nx)$ on $[0,\pi]$, does the sequence $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence? Prove or disprove it.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative continuous functions on [0,1] such that $f_n(x) \geq f_{n+1}(x)$, $n=1,2,\cdots$, for all $x \in [0,1]$. Let $f(x)=\lim_{n\to\infty} f_n(x)$ and $M=\sup_{0\leq x\leq 1} f(x)$, show that there is a $t \in [0,1]$ such that $f(t)=M$.

5. Let $f:\mathbb{R}\to\mathbb{R}$ be a function.

(1) If f is continuous in the ϵ-δ sense, show that $f^{-1}(C)$ is closed for any closed subset C of $\mathbb{R}$.

(2) If $f^{-1}(K)$ is compact for any compact subset K of $\mathbb{R}$, is f continuous on $\mathbb{R}$? Prove it or give a counterexample.

6. Define

$$f(x,y)=\begin{cases}(x-y)^2\sin\frac{1}{x-y}, & \text{if } x\neq y,\\ 0, & \text{if } x=y.\end{cases}$$

Is f differentiable at (0,0)? Prove or disprove it.

7. Let a,b be two positive real numbers. Evaluate

$$\lim_{k\to\infty}\left(\frac{a^{\frac{1}{k}}+b^{\frac{1}{k}}}{2}\right)^k.$$

8. Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms, and let $L_k=\frac{\ln\frac{1}{a_k}}{\ln k}$. Show that if $\lim_{k\to\infty} L_k > 1$, then the series converges.