國立清華大學命題紙

九十二學年度 數學 系（所） 應用數學 組碩士班研究生招生考試

科目 線性代數 科號 0202 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1.(15%) Find the value of c so that the system of linear equations $\begin{cases} x+y+z=1 \\ x-y+z=6 \\ x+5y+z=c \end{cases}$ has solutions in $\mathbb{R}^3$, and in that case, find all the solutions.

2.(15%) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 5 \end{bmatrix}$.

(a) Find a nonsingular matrix P such that $PA = B$.

(b) Is P in (a) unique? Give reasons.

3.(15%) (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that is the reflection with respect to the plane $\{(x,y,z) : x+y-2z=0\}$. Find the matrix representation of T with respect to the basis $\{e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1)\}$ of $\mathbb{R}^3$.

(b) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to the xy-plane, and put $A = ST$. Find a line L passing through the origin such that A leaves L pointwise fixed.

4.(15%) (a) Let A be a $m \times n$ real matrix, B a $n \times p$ real matrix, prove that $rank(AB) \geq rankA + rankB - n$.

(b) Use (a) to show that if $A_1, \ldots, A_k$ are $n \times n$ real matrices satisfying $A_1 \cdots A_k = 0$, then $rankA_1 + \ldots + rankA_k \leq (k-1)n$.

5.(15%) Let $A = (a_{ij})$ be a real 3×3 matrix, and $B = (b_{ij})$ the transpose matrix of the corresponding cofactors , that is, $b_{ij} = (-1)^{i+j} \det A_{ji}$ where A_{ij} is the 2×2 matrix obtained from A by deleting its ith row and jth column. Prove that

(a) if $rankA = 3$, then $rankB = 3$;

(b) if $rankA = 2$, then $rankB = 1$.

6.(25%) For each of the following statements, sketch a proof if it is true, explain why or give a counterexample if it is false.

(a) If a system of linear equations with integral coefficients has real solutions, then there exists rational solutions for the same system.

(b) Let A, B be real symmetric $n \times n$ matrices, then there exists a nonsingular matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

(c) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation satisfying $T^4 = -I$, then n has to be even.

(d) If A is a singular $n \times n$ real matrix, then there exists a nonzero $n \times n$ matrix B satisfying $BA = 0$.

(e) If $a < 0$, then $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$ is diagonalizable over $\mathbb{R}$.