九十學年度 物理 系（所） 組碩士班研究生招生考試

應用數學 科號 0603 共 二 頁第 一 頁 *請在試卷【答案卷】內作答

1 (20%)

A Hermitian matrix H is

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

(a) Find the eigenvalues of H.

(b) Find the normalized eigenvectors of H.

(c) Prove these eigenvectors are orthogonal to each other and complete.

(d) Find the unitary matrix which diagonalizes H.

(e) Find the inversion of H.

2 (10%)

Find the explicit expression of $M = e^{\frac{i}{2}\theta\sigma}$ where

$$\sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} .$$

(Hint: $e^A \equiv 1 + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$)

3 (20%)

Compute the volume V of a n-dimensional sphere of radius R, i.e., compute the integral

$$V = \int dx_1 dx_2 \cdots dx_n$$

over the domain of $0 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le R^2$, where $n = 2N$ (i.e. n is an even number).

4 (10%)

Evaluate in closed form the sum $S(x) = \sum_{n=1}^{\infty} x^n n^2$ for $|x| < 1$.

5 (20%)

Evaluate the integrals:

$$(a) \quad I_a = \int_{-\infty}^{+\infty} x^4 e^{-\alpha x^2} dx$$

(Hint: $\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$);

$$(b) \quad I_b = \int_0^{2\pi} \frac{d\theta}{\lambda + \cos\theta}$$

with $\lambda > 1$.

6 (20%)

Consider the inhomogeneous differential equation

$$f''(x) + 2z f'(x) + k^2 f(x) = \delta(x - x_0)$$

where k and $z > 0$ are real constants and $\delta(x)$ is the Dirac δ- function.

Find the general solution for $k^2 > z^2$.