1 (10%)

Find the both Fourier *cosine* and *sine* series of the Dirac delta function $\delta(x-x')$ in the interval $[0, L]$.

2 (20%)

The Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find

(a) the commutator $[\sigma_i, \sigma_j] \equiv \sigma_i\sigma_j - \sigma_j\sigma_i$,

(b) the anticommutator $\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i$,

(c) $Tr(\vec{\sigma} \cdot \vec{a})$,

(d) $Tr[(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})]$,

(e) $Tr[(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})(\vec{\sigma} \cdot \vec{c})]$,

where $i, j = 1, 2, 3$ and $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} - (c_1, c_2, c_3)$ are three constant vectors.

3 (20%)

The n-dimensional volume V_n is given by

$$V_n = \int dx_1 dx_2 \cdots dx_n .$$

Compute the volumes of (a) $n = 5$ and (b) $n = \infty$-dimensional unit spheres.

4 (20%)

Calculate the integrals:

(a) $$I_a = \int_0^{2\pi} d\phi \frac{b + a\cos\phi}{a^2 + b^2 + 2ab\cos\phi}, \quad with \ |b| > |a|,$$

(b) $$I_b = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - a^2 - i\epsilon)^3}, \quad with \ a > 0.$$

5 (20%)

Consider the motion of a damped harmonic oscillator under the action of an external force. The differential equation reads

$$m\ddot{x} + \rho\dot{x} + kx = I\,\delta(t),$$

where I, m, ρ, and k are positive constants and $\delta(t)$ is the Dirac delta function. Solve the equation with the initial conditions:

$$x(0) = 0, \quad \dot{x}(0) = 0.$$

6 (10%)

If n is a positive integer and $x - n \neq 0, -1, -2\cdots$,

evaluate

$$\frac{\Gamma(x+n)}{\Gamma(x-n)},$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$ is the Gamma function.