1. Assume A is a 3×3 matrix given by

$$A=\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}.$$

(a) Find the eigenvalues λ_i and their corresponding eigenvectors. (10%)

(b) Compute $\sum_{i=1}^{3}\lambda_i$ and $\prod_{i=1}^{3}\lambda_i$. (5%)

(c) Find $A^{\frac{1}{2}}$ for which $A=A^{\frac{1}{2}}A^{\frac{1}{2}}$. (10%)

2. Consider the system of equations $A\underset{\sim}{x}=\underset{\sim}{c}$. The least squares solution of the system is the solution of $\underset{\sim}{x}$ minimizing $(A\underset{\sim}{x}-\underset{\sim}{c})'(A\underset{\sim}{x}-\underset{\sim}{c})$.

(a) Show that $\underset{\sim}{\hat{x}}$ is a least squares solution if $A'A\underset{\sim}{\hat{x}}=A'\underset{\sim}{c}$. (10%)

(b) Assume $A=\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$.

Find a least squares solution for $A\underset{\sim}{x}=\underset{\sim}{c}$. (5%)

3. Let A be an $m\times m$ idempotent matrix.

(a) Give the definition of idempotent matrix. (2%)

(b) Prove that I-A is also idempotent where I is the m×m identity matrix. (3%)

(c) Prove that each eigenvalue of A is 0 or 1. (5%)

4. Compute the volumn of the ellipsoid:

$$\left(\frac{x-\mu_1}{a}\right)^2 + \left(\frac{y-\mu_2}{b}\right)^2 + \left(\frac{z-\mu_3}{c}\right)^2 = 1 \qquad (10\%)$$

5. Evaluate:

(a) $\int_0^1 \ln(1+x)dx$. (5%)

(b) $\int \frac{1}{x\sqrt{x^2-1}}dx$. (5%)

6. 敘述微積分基本定理 （5%）

7. Find the Taylor expansion of $f(x) = \ln\frac{1+x}{1-x}$ for $|x|<1$. (10%)

(Hint: Find the expansions for ln(1-x) and ln(1+x) separately first and then combine them.)

8. For the following series, find the limit if the series converge or given a reason if the series diverge. (15%)

(a) $\sum_{n=1}^{\infty} n^2 e^{-n}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$