1. (15%)
Which of the following statements are true?
(a) There is a one-to-one correspondence between the set of positive rational numbers and the set of natural numbers.
(b) The complement of a spanning tree contains a cut-set.
(c) The language $L = \{1^i0^j | i \geq j\}$ is a finite state language.
(d) A nondeterministic finite state machine is not more powerful than a deterministic finite state machine.
(e) It takes exponential number of steps to solve any NP-complete problem in the worst case.

2. (8%)
Given two multisets **P** = {*a, a, a, c, d, d*} and **Q** = {*a, a, b, c, c*}, answer the following questions:

$\mathbf{P} \cup \mathbf{Q} =$ ＿＿＿, $\mathbf{P} \cap \mathbf{Q} =$ ＿＿＿,
$\mathbf{P} - \mathbf{Q} =$ ＿＿＿, $\mathbf{P} + \mathbf{Q} =$ ＿＿＿.

3. (6%)
Explain the definitions of the following two types of grammar:
(a) Type-2 grammar,
(b) Type-0 grammar.

4. (8%)
Let $A = \{a, b, c, d, e, f, g, h, i, j, k\}$,

$$\pi_1 = \{\overline{abcd}, \overline{efg}, \overline{hi}, \overline{jk}\}, \pi_2 = \{\overline{abch}, \overline{di}, \overline{efjk}, \overline{g}\}$$

be two partitions of A.
Let the product $\pi_1 \bullet \pi_2$ of π_1 and π_2 to be the partition corresponding to the equivalence relation $R_1 \cap R_2$, and let the sum $\pi_1 + \pi_2$ of π_1 and π_2 to be the partition corresponding to the equivalence relation $R_1 \cup R_2$. Compute $\pi_1 \bullet \pi_2$ and $\pi_1 + \pi_2$.

5. (6%)
Consider the finite state machine shown below. If the input sequence is 1122212212, what's the output sequence will be?

State	Input		Output
	1	2	
⇒ A	B	C	0
B	C	D	0
C	D	E	0
D	E	B	0
E	B	C	1

6. (6%)

Construct an optimum binary tree for the weights 5, 6, 7, and 12.

A binary tree T for the weights $w_1, w_2, \ldots, w_t$ is said to be an *optimal tree* if,

$$\sum_{i=1}^{t} w_i l(w_i)$$

is minimum, where $l(w_i)$ is the path length of the leaf to which the weight w_i is assigned.

7. (10%)

Prove that $I[A, B] = I[B, A]$, where A and B are two events, $I[A, B]$ is denoted as the mutual information from B to A.

8. (6%)

There is a barber in a small village. He claims that he will shave everybody who does not shave himself. Show that there is no such barber can exist.

9. (5%)

If G is a graph with v vertices and e edges, how many edges can be removed without causing the remaining graph to be disconnected.

10. (10%)

(a) Prove that the generating function to determine the number of partitions of the integer n into the integers 1, 2, …, k with repetitions allowed is

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^k)}.$$

(b) Use the generating function to prove that the number of partitions of the integer n into odd integers with repetitions allowed equals to the number of partitions of the integer n into distinct parts.

11. (10%)

(a) Prove that in any connected planar graph $G = (V, E)$ with at least three vertices, $3v - e \geq 6$, where $|V| = v$ and $|E| = e$.

(b) Use (a) to prove that every planar graph G contains a vertex of degree 5 or less.

12. (10%)

Let a_n denote the number of ways that a person climb up a ladder on n rungs (階梯) if at each step he can climb either one or two rungs.

(a) Derive a recurrence relation for a_n in terms of a_{n-1} and a_{n-2}.

(b) Solve the recurrence relation.