1. (10%) If matrix $\boldsymbol{A}$ is similar to matrix $\boldsymbol{B}$, show that:
   (a) (5%) $\det(\boldsymbol{A}^c - c\,\boldsymbol{I}) = \det(\boldsymbol{B}^c - c\,\boldsymbol{I})$ for some scalar c.
   (b) (5%) The trace of $\boldsymbol{A}^T$ is equal to that of $\boldsymbol{B}^T$.

2. (15%) Suppose a linear operator L transforms (1, 0, -1) to (0, 0, -2), (1, -1, 2) to (-1, 7, 1), and (-1, -1, 1) to (-1, 1, 2), respectively.
   (a) (5%) Find the matrix $\boldsymbol{A}$ that represents L.
   (b) (5%) Find the kernel of L.
   (c) (5%) Find the determinant of the *adjoint* of matrix $\boldsymbol{A}$, namely, det(adj $\boldsymbol{A}$).

3. (5%) Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

4. (12%) Suppose matrix $\boldsymbol{A}$ equals $\boldsymbol{L}\times\boldsymbol{U}$, where $\boldsymbol{L}$ is a unit lower triangular matrix, and $\boldsymbol{U}$ is a unit upper triangular matrix. Prove or disprove the following:
   (a) (3%) $\boldsymbol{A}$ shares the same row space with L.
   (b) (3%) $\boldsymbol{A}$ shares the same column space with L.
   (c) (3%) $\boldsymbol{A}$ shares the same row space with U.
   (d) (3%) $\boldsymbol{A}$ shares the same column space with U.

5. (8%) Given $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$, and the linear transformation L from R^2 into R^3 defined as: $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1\,\mathbf{b}_2 + x_2\,\mathbf{b}_1 + (x_1 + x_2)\,\mathbf{b}_3$.
   (a) (2%) Find the matrix $\boldsymbol{A}$ representing L with respect to the bases $[\mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$.
   (b) (2%) Find the matrix $\boldsymbol{B}$ representing L with respect to the bases $[\mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$.
   (c) (2%) Find the matrix $\boldsymbol{C}$ representing L with respect to the bases $[\mathbf{a}_1, \mathbf{a}_2]$ and $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$.
   (d) (2%) Find the matrix $\boldsymbol{D}$ representing L with respect to the bases $[\mathbf{a}_1, \mathbf{a}_2]$ and $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$.

6. (15%) Let $S = Span\left((1\ \ 3\ \ 1\ \ 1)^T, (1\ \ 1\ \ 1\ \ 1)^T, (-1\ \ 5\ \ 2\ \ 2)^T\right)$ be a subspace of $\mathbf{R}^4$, and let $\mathbf{b} = (4\ \ -1\ \ 5\ \ 1)^T$.
   (a) (5%) Find an orthonormal basis for S.
   (b) (5%) Use your answer in (a) to find the projection $\mathbf{p}$ of $\mathbf{b}$ onto S.
   (c) (5%) Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 5 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

   Use your answer in (b) to solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

7. (18%) Let A be a diagonalizable matrix.
   (a) (6%) Show that the number of nonzero eigenvalues (counted according to multiplicity) of A equals the rank of A.
   (b) (6%) Show that e^A is nonsingular.
   (c) (6%) Let $p(\lambda)$ be the characteristic polynomial of A:

   $$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0.$$

   Show that $p(A) = 0$.

8. (5%) Let A be a symmetric $n\times n$ matrix. Show that e^A is symmetric and positive definite.

9. (12%) Give

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

   (a) (6%) Find the Cholesky decomposition LL^T of A, where L is lower triangular with positive diagonal elements.
   (b) (6%) Find a unitary matrix U that diagonalizes A.