(50%) I. Answer the following questions.

1. Let the number of changes that occur in a given continuous interval be counted. We have an (approximate) Poisson process with parameter $\lambda > 0$ if the following statements are satisfied. Please fill in the blanks.
   (a) (2%) The numbers of changes occurring in non-overlapping intervals are stochastically ____________.
   (b) (2%) The probability of exactly one change in a sufficiently short interval of length h is approximately ______.
   (c) (2%) The probability of two or more changes in a sufficiently short interval is essentially________.

2. In an (approximate) Poisson process with mean λ, answer the following questions:
   (a) (3%) The waiting time until the first change has an _________distribution.
   (b) (3%) The waiting time until the αth change has a _______ distribution.

3. (6%) A drawer contains eight pairs of socks. If six socks are taken at random and without replacement, compute the probability that there is at least one matching pair among these six socks.

4. (7%) A hospital obtains 40% of its flu vaccine from Company A, 50% from Company B, and 10% from Company C. From past experience it is known that 3% of the vials from A are ineffective, 2% from B are ineffective, and 5% from C are ineffective. The hospital tests five vials from each shipment. If at least one of the five is ineffective, find the conditional probability of that shipment coming from C.

5. Let X and Y have the joint p.d.f. $f(x, y) = e^{-2}/[x!(y-x)!]$, $y = 0, 1, \ldots; x = 0, 1, \ldots, y$, zero elsewhere.
   (a) (5%) Find the moment-generating function $M(t_1, t_2)$ of this joint distribution.
   (b) (5%) Compute the means, the variances, and the correlation coefficient of X and Y.
   (c) (4%) Determine the condition mean $E(X|y)$.

6. (6%) Suppose that a woman leaves for work between 8:00 a.m. and 8:30 a.m. and takes between 40 and 50 minutes to get to the office. Let X denote the time of departure and let Y denote the time of travel. If we assume that these random variables are stochastically independent and uniformly distributed, find the probability that she arrives at the office before 9:00 a.m.

7. (5%) Let X have the distribution $F(x)$ of the continuous type that is strictly increasing on the support $a < x < b$. Prove that the random variable Y, define by $Y = F(X)$, has a distribution that is U(0, 1).

(50%) **II. Answer the following questions.**

1. (10%) Let the joint probability mass function of X and Y be

$$f(x,y) = \frac{1}{4}, \quad (x,y) \in S = \{(0,0),(1,1),(1,-1),(2,0)\}$$

   (a) Are X and Y independent?

   (b) Calculate the covariance $\mathrm{Cov}(X,Y)$ and the correlation coefficient $\rho(X,Y)$.

2. (10%) Let Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a Poisson distribution with mean $\lambda = 3$.

   (a) Find the mean and variance of $Y = X_1 + X_2 + \cdots + X_n$.

   (b) Find the moment generating function of Y, $E[e^{tY}]$.

3. (10%) Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad 0 < x < \infty$.

   (a) Show that $\Gamma(x+1) = x\Gamma(x) \quad \forall\, x > 0$.

   (b) Calculate $\Gamma(\frac{5}{2})$.

4. (20%) If the p.d.f. of X is

$$f(x) = \frac{1}{\sqrt{32\pi}} exp\left[-\frac{(x+3)^2}{32}\right], \quad -\infty < x < \infty$$

   (a) Find the mean μ and variance σ^2 of X.

   (b) Find the moment generating function $M_X(t)$ of X.

   (c) Define $Y = \frac{X+3}{4}$, what is the distribution of Y?

   (d) Define $Z = Y^2$, what is the distribution of Z?