1. (5%)$\mathcal{L}$ represents the Laplace Transform operator.

$$\mathcal{L}\Big(t\cos(2t)\Big) = (\underline{\ \ (1)\ \ } - 4)\cdot \underline{\ \ (2)\ \ }$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A) s^2; (B) s^{-2}; (C) $(s-1)$; (D) $(s-1)^2$; (E) $(s-1)^{-2}$; (F) (s^2-1); (G) (s^2-2); (H) $(s-2)^2$; (I) $(s-2)^{-2}$; (J) $(s^2+4)^{-2}$; (K) $(s^2+4)^2$.

2. (5%)$\mathcal{L}^{-1}$ represents the inverse Laplace Transform operator.

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \underline{\ \ (1)\ \ }\Big(\sin(\omega t) - \underline{\ \ (2)\ \ }\cdot\cos(\omega t)\Big)$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A)2ω; (B)$\frac{1}{2\omega}$; (C)$2\omega^2$; (D)$\frac{1}{2\omega^2}$; (E)$2\omega^3$; (F)$\frac{1}{2\omega^3}$; (G)ωt; (H)$(\omega t)^2$; (I)$(\omega t)^3$; (J)$2\omega^2 t$; (K)$2\omega^2 t$.

3. (5%)'$*$' represents the convolution operator.

$$(e^{-t} - e^{-2t}) * e^{-t} = \underline{\ \ (1)\ \ } + (t-1)\underline{\ \ (2)\ \ }$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A)e^t; (B)$e^{(t-1)}$; (C)e^{-t}; (D)$e^{-(t-1)}$; (E)e^{-2t}; (F)$e^{-2(t-1)}$; (G)e^{2t}; (H)$e^{2(t-1)}$; (I)e^{-3t}; (J)$e^{-3(t-1)}$; (K)e^{3t}; (L)$e^{3(t-1)}$; (M)t; (N)$(t-1)$; (O)$\frac{1}{t-1}$; (P)$(t-1)^2$; (Q)$(t-1)^3$.

4. (5%)Please identify all the even functions in the following. Full grade will be given only if all answers are correct. (A)e^x; (B)$e^{(x^2)}$; (C)$\sin(nx)$; (D)$x\sin(x)$; (E)$\frac{\cos(x)}{x}$; (F)$ln(x)$; (G)$\sin(x^2)$; (H)$\sin^2(x)$.

5. (5%)Which of the following collections of vectors are linearly independent in R^3? R^3 represents a Euclidean vector space. (A)$(1,0,0)^T, (0,1,1)^T, (1,0,1)^T$; (B)$(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T$; (C)$(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T$; (D)$(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T$;(E)$(1,1,3)^T, (0,2,1)^T$.

6. (7%) The standard 2nd-order mass-damper-spring system can be expressed by the differential equation $m\ddot{x} + b\dot{x} + kx = F(t)$, where x is the displacement of the proof mass, b is the damping coefficient, k is the spring constant, and $F(t)$ is the externally applied force. The equation can be re-written in another form as $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = F(t)/m$, where ξ is the damping ratio and ω_n is the natural frequency defined as $\omega_n = \sqrt{\frac{k}{m}}$. Now that the applied force $F(t)$ is a unit-step function $u(t)$ and $0 < \xi < 1$. Determine the corresponding particular solution from the following answers (note: $\omega_d = \omega_n\sqrt{1-\xi^2}$):

(1) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}(\cos\omega_d t + \sin\omega_d t)\right]$

(2) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}(1 + \omega_n t)\right]$

(3) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}\left(\frac{1}{\sqrt{1-\xi^2}} + \omega_n t\right)\right]$

(4) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t} - e^{-\xi\omega_d t}\right]$

(5) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t} - \frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_d t}\right]$

(6) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$

(7) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}\left(\frac{1}{\sqrt{1-\xi^2}}\cos\omega_d t + \sin\omega_d t\right)\right]$

(8) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\cos\omega_d t + \sin\omega_d t\right)\right]$

(9) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}(c_1\cos\omega_d t + c_2\sin\omega_d t)\right]$, c_1 and c_2 are arbitrary constants.

(10) $x(t) = \frac{1}{k}\left[1 - e^{-\xi\omega_n t}(c_1 + c_2\omega_n t)\right]$, c_1 and c_2 are arbitrary constants.

7. (6%) The differential equation $axy'' + y' + y = 0$ ($0 < x < \infty$, a is an unknown constant) has two linear independent solutions expressed in power series: $y_1(x) = 1 - x + \frac{x^2}{8} - \frac{x^3}{168} + \frac{x^4}{6720} - \dots$, $y_2(x) = x^{2/3} - \frac{x^{5/3}}{5} + \frac{x^{8/3}}{80} - \frac{x^{11/3}}{2640} + \dots$.
Please determine the value of a that leads to these two solutions. (1) a = -1 (2) a = 1 (3) a = -2 (4) a = 2 (5) a = -3 (6) a = 3 (7) a = -4 (8) a = 4 (9) a = -1/2 (10) a = 1/2.

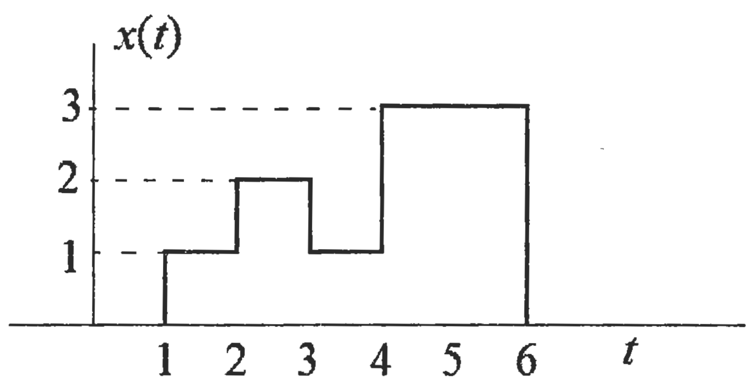
8. (7%) Determine the general solution of the differential equation $y' = y^2 - xy + 1$, which has a particular solution $Y(x) = x$ by inspection (note: C is an arbitrary constant).

(1) $y(x) = x + \frac{2e^{-x^2/2}}{C - 3\int e^{x^2/2}dx}$ (2) $y(x) = x + \frac{e^{x^2/2}}{C - \int e^{x^2/2}dx}$ (3) $y(x) = x + \frac{e^{-x^2/2}}{C - \int e^{-x^2/2}dx}$ (4) $y(x) = x + \frac{2e^{x^2/2}}{C + \int e^{-x^2/2}dx}$ (5) $y(x) = x - \frac{e^{-x^2/2}}{C - 2\int e^{x^2/2}dx}$

(6) $y(x) = x - \frac{e^{x}}{C - \int e^{x}dx}$ (7) $y(x) = x + \frac{2e^{x}}{C + \int e^{-x}dx}$ (8) $y(x) = x + \frac{2e^{x}}{C + 3\int e^{-x}dx}$

(9) $y(x) = x + \frac{e^{-x}}{C + \int e^{-x}dx}$ (10) $y(x) = x + \frac{2e^{x}}{C + 2\int e^{-x}dx}$.

9. (7%) Find the Fourier transform of the function $x(t)$ shown below.



10. (10%) Solve for u(x,t) that satisfies $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the following conditions

$u(0,t) = u(1,t) = 0$ for all t

$u(x,0) = \sum_{n=1}^{7} \frac{1}{n} \sin n\pi x$, $\frac{\partial u}{\partial t}\Big|_{t=0} = 0$ for $0 < x < 1$

You need to show how you derive your answer. Partial points will be deducted for not writing your derivation.

11. (4%) Find a scalar function $f(x,y,z)$ such that $\nabla f = 6x\vec{i} + 2\vec{j} + 2z\vec{k}$. No need to write down the derivation. Just giving your answer is OK.

(12%) Then, choose an answer for each of the following integrals along the specified paths: (No need to write down the derivation. Just pick up the correct value for each integral.)

$\int_C (6x\vec{i} + 2\vec{j} + 2z\vec{k}) \bullet d\vec{r}$ =(a) 0 (b) 2 (c) 2π (d) 4π (e) 4 (f) 6 (g) 2.5π (h) 10 (i) none of the above

$\int_D (6x\vec{i} + 2\vec{j} + 2z\vec{k}) \bullet d\vec{r}$ =(a) 0 (b) 2 (c) 2π (d) 4π (e) 4 (f) 6 (g) 2.5π (h) 8 (i) none of the above

The absolute value of $\oint_E (yz\vec{i} + 6xz^5\vec{j} - xy^2z\vec{k}) \bullet d\vec{r}$ is equal to (a) 0 (b) 2π (c) 3π (d) 5π (e) 8π (f) 11π (g) 13π (h) π (i) none of the above

Here, C is the path from the point (0,0,0) to (1,1,1) following a straight-line segment.

D is the path first from the point (0,0,0) to $(0,\frac{1}{2},0)$ following a straight-line segment, and then from $(0,\frac{1}{2},0)$ to (1,1,1) again following a straight-line segment.

E is the path along the circle: $x^2 + y^2 = 1, \quad z = 1$.

12. (12%) $\int_0^\infty \frac{\sin x}{x} dx =$ (a) 0.4π (b) π (c) 2π-4 (d) 2.5π-5 (e) 0.6π (f) 0.8π

(g) π-2 (h) 0.5π (i) none of the above. (You may use the residue theorem.)

13. (10%) Evaluate the integrals along the path C that is the counterclockwise circle with $|z| = 3$.

(a) $\oint_C \frac{z^2 - 1}{z^2 + 1} e^z dz$

(b) $\oint_C \frac{\sinh 3z}{(z^2 + 1)^2} dz$