國 立 清 華 大 學 命 題 紙

95 學年度 電機領域聯合招生 系（所） ______ 組碩士班入學考試

科目 工程數學 A 科目代碼 9902 共 5 頁第 1 頁 *請在【答案卷卡】內作答

For problems 1~5, both correct answers and detailed works are required.

1. (5 %) Find the sine half-range expansion of $f(x)$

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & \dfrac{L}{2} < x < L \end{cases}$$

(A) $\frac{4k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x + \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x + \ldots\right)$ (B) $\frac{4k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{2^2}\sin\frac{2\pi}{L}x + \frac{1}{3^2}\sin\frac{3\pi}{L}x - \ldots\right)$

(C) $\frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - \ldots\right)$ (D) $\frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x + \frac{1}{2^2}\sin\frac{2\pi}{L}x + \frac{1}{3^2}\sin\frac{3\pi}{L}x + \ldots\right)$

(E) $\frac{4k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{2\pi}{L}x + \frac{1}{5^2}\sin\frac{3\pi}{L}x - \ldots\right)$ (F) $\frac{6k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{2\pi}{L}x + \frac{1}{5^2}\sin\frac{3\pi}{L}x - \ldots\right)$

(G) $\frac{2k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{2\pi}{L}x + \frac{1}{5^2}\sin\frac{3\pi}{L}x - \ldots\right)$ (H) none of the above

2. (5 %) Find the Fourier transform of $f(x)$

$$f(x) = e^{-|x+3|} - 2e^{-|x|}$$

(A) $\frac{1}{\sqrt{2\pi}(w+1)}\left(e^{-i3w}-2\right)$ (B) $\frac{2}{\sqrt{2\pi}(w+1)}\left(e^{i3w}-2\right)$ (C) $\frac{2}{\sqrt{2\pi}(w^2+1)}\left(e^{-i3w}-2\right)$

(D) $\frac{2}{\sqrt{2\pi}(w^2+1)}\left(e^{i3w}-2\right)$ (E) $\frac{1}{\sqrt{2\pi}(w^2-1)}\left(e^{i3w}-2\right)$ (F) $\frac{1}{\sqrt{2\pi}(w^2+1)}\left(e^{i3w}-2\right)$

(G) $\frac{1}{\sqrt{2\pi}(w^2-1)}\left(e^{-i2w}-3\right)$ (H) none of the above

3. (5 %) Find the inverse Laplace transform of

$$F(s)=\frac{1}{s(s^2+\omega^2)}$$

(A) $\frac{1}{w^2}(1-\sin wt)$ (B) $\frac{1}{w^2}(1+\cos wt)$ (C) $\frac{1}{w^2}(1-\cos wt)$ (D) $\frac{1}{w}(1-\sin wt)$

(E) $\frac{1}{w}(1+\cos wt)$ (F) $\frac{1}{w}(1+\tan wt)$ (G) $\frac{1}{w}(1-\tan wt)$ (H) none of the above

4. (10 %) Use Laplace transform to solve

$$xy''+(1-x)y'+ky=0$$

(A) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^{-k}e^{-t}]$ (B) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^k e^t]$ (C) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^k e^{-t}]$ (D) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^k e^{-t}]$

(E) $y=\frac{e^{-t}}{k!}\frac{d^k}{dt^k}[t^k e^{-t}]$ (F) $y=\frac{e^{-t}}{k}\frac{d^k}{dt^k}[t^k e^{-t}]$ (G) $y=\frac{e^k}{t!}\frac{d^k}{dt^k}[t^k e^{-t}]$ (H) none of the above

5. (10 %) Use Method of Frobenius to solve the general solution of

$$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$$

(A) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n-\frac{1}{2}}$ (B) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}$

(C) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}$ (D) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{n+\frac{1}{2}}$

(E) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)!}x^{n-\frac{1}{2}}$ (F) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n-1)!}x^{n-\frac{1}{2}}$

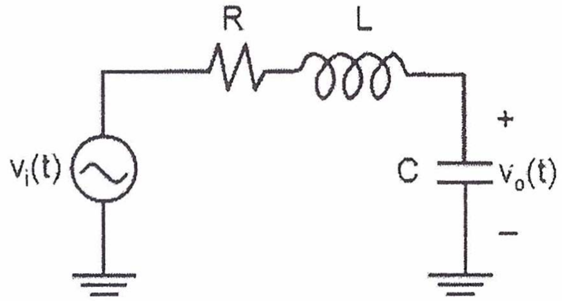
(G) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{}{(2n-1)!}x^{n+\frac{1}{2}}$ (H) none of the above

(c_1 and c_2 are arbitrary constants)

6. (a) (3%)The R-L-C network as shown has a sinusoidal input $v_i(t) = \sin(\omega_o t)$, and the output voltage across the capacitor is described by the differential equation:

$$\frac{d^2 v_o(t)}{dt^2} + 30\frac{dv_o(t)}{dt} + 22500 v_o(t) = v_i(t)$$

where the coefficients are determined by the value of each passive component.



You are required to calculate the input frequency ω_o that will cause the output $v_o(t)$ to have an exact 90° phase delay with respect to the input $v_i(t)$, as the output reaches its steady state (namely, the particular solution of the differential equation).

(b)(4%) By using the differential operator $D^n = \frac{d^n}{dx^n}$, the differential equation

$$\frac{d^6 y}{dx^6} + 2\frac{d^5 y}{dx^5} + 9\frac{d^4 y}{dx^4} - 2\frac{d^3 y}{dx^3} - 10\frac{d^2 y}{dx^2} = sin(3x) + 3x^2 + xe^{-x}$$

is re-written as

$$(D^2 + 2D + 10)(D^4 - D^2)y = sin(3x) + 3x^2 + xe^{-x}.$$

Please determine the correct representation of the particular solution y_p for solving, and you do not have to solve the coefficients in it.

7. (5%) Solve the differential equation $cos\,x \cdot dx + (sin\,x + cos\,y - sin\,y) \cdot dy = 0$.

8. (8%) Solve the differential equation $x^3 \frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} - 9xy = 1 \quad (x > 0)$.

9. (10%) Evaluate the integral $\oint_C e^{\frac{1}{z^2}} dz$ where C: $|z|=4$ **counterclockwise.**

10. (10%) Find the eigenvalues and corresponding normalized eigenvectors (norm equals to 1) for the matrix $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 4 & 2 & 5 \end{bmatrix}$.

11. The position $\bar{r}$ of a particle of mass $m=1$ at time t is described as (all physical quantities are in SI units):

$$C: \ \bar{r}(t) = \frac{t^2}{\sqrt{2}}\bar{i} + (t+1)\bar{j} + \frac{t^3}{3}\bar{k}, \ t=[0,1].$$

(a) (4%) Let V and W denote the average speed (a scalar) and work done to move the particle from t=0 to t=1, respectively. Choose the correct answer of (V,W) from the following:

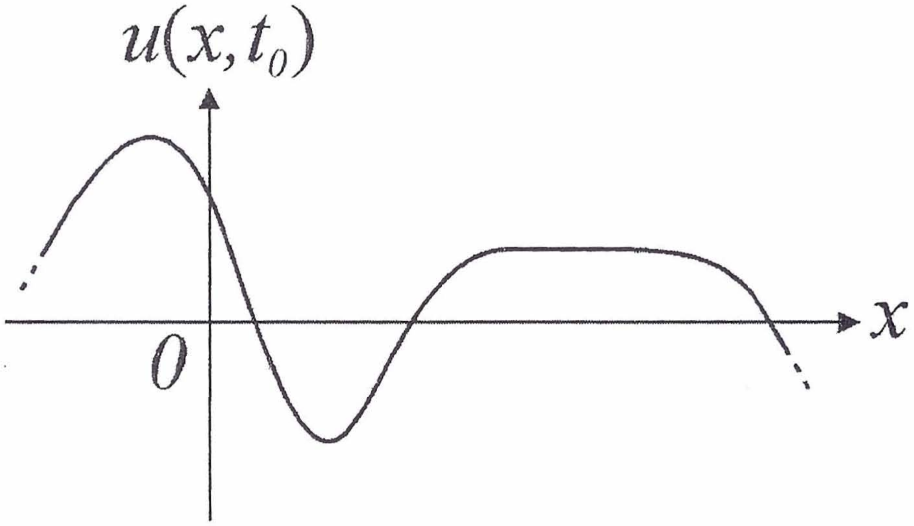
(a) (1,2); (b) (2,1); (c) $\left(\frac{1}{3},\frac{1}{2}\right)$; (d) $\left(\frac{1}{2},\frac{2}{3}\right)$; (e) $\left(\frac{3}{2},\frac{4}{3}\right)$; (f) $\left(\frac{4}{5},\frac{3}{2}\right)$; (g) (1,1); (h) $\left(1,\frac{1}{2}\right)$; (i) $\left(\frac{4}{3},\frac{5}{2}\right)$; (j) $\left(\frac{1}{2},\frac{1}{3}\right)$; (k) $\left(\frac{4}{3},\frac{3}{2}\right)$; ($l$) none of the above.

(b) (3%) If there exists an electric field $\bar{E}(x,y,z) = y\cdot\cos(z)\bar{i} + x\cdot\cos(z)\bar{j} - xy\cdot\sin(z)\bar{k}$. What is the work W_E done by the field $\bar{E}$ to move the particle of charge $q=\sqrt{2}$ along the specified path C: $\bar{r}(t)$, t=[0,1]?

(a) $\sin(2)$; (b) 1; (c) $\sin\left(\frac{1}{3}\right)$; (d) $2\sin\left(\frac{2}{3}\right)$; (e) $\sqrt{2}\cos\left(\frac{1}{3}\right)$; (f) $\sqrt{2}\sin\left(\frac{2}{3}\right)$; (g) $\sqrt{2}$; (h) $\frac{\sqrt{3}}{2}$; (i) $\frac{2}{3}\cos\left(\frac{2}{3}\right)$; (j) $\frac{1}{2}\cos\left(\frac{1}{3}\right)$; (k) $2\cos\left(\frac{1}{3}\right)$; ($l$) none of the above.

12. The motion of a string is governed by the partial differential equation (PDE): $u_{tt}=c^2u_{xx}$; where $u(x,t)$ is the displacement of the particle at position x and time t, c is a real constant, the subscripts tt, xx denote $\partial^2/\partial t^2$, $\partial^2/\partial x^2$, respectively.

(a) (5%) The following figure shows a section of the string at some instant $t=t_0$, please roughly sketch the force vectors imposing on the illustrated string section.



(b) (8%) Let the string has a finite length L ($0\le x\le L$), and the two ends slide vertically without friction, i.e. boundary conditions (BCs) are: $u_x(0,t)=u_x(L,t)=0$, where the subscript x denotes $\partial/\partial x$. One can derive discrete modes $u_n(x,t)=X_n(x)\cdot T_n(t)$ (functions satisfying the PDE and BCs) by using the method of separation of variables. Please sketch the spatial profile $X_n(x)$ for the lowest three (nontrivial) modes.

(c) (5%) In the presence of initial conditions (ICs): $u(x,0)=f(x)$, $u_t(x,0)=g(x)$, one usually expands the solution in terms of the modes: $u(x,t)=\sum_n \{A_n\}u_n(x,t)$, where $\{A_n\}$ is(are) the coefficient(s) for mode $u_n(x,t)$, then substitutes ICs to retrieve $\{A_n\}$. Although the principle of superposition works for the PDE of this problem ($u_{tt}=c^2u_{xx}$), it could fail in some other PDEs. Please specify those of the following PDEs for which superposition does NOT apply.

(a) $u_{tt}=p(x)\cdot u_{xx}$; (b) $u_{tt}=p(x)\cdot u_{xx}+q(x,t)$; (c) $u_{tt}=u_{xx}+u_{xt}$; (d) $u_{tt}=p^2(x)\cdot u_{xx}+u_{xt}$; (e) $u_{tt}= u\cdot u_t+u_x$; (f) $u_t=\exp[u_x]+u_{tt}$; (g) $u_{ttx}=p(x,t)\cdot u_{xxt}$; (h) $u_{tt}= \exp[p(x,t)]\cdot u_{xx}+u$.