1. Solve the following ordinary differential equations

(1a) $y' = 2 + \sqrt{y - 2x + 3}$ (7%)

(1b) $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$ (8%)

2. Solve the following pair of simultaneous differential equations by Laplace transform

$$\frac{d^2x}{dt^2} - x = y$$

$$\frac{d^2y}{dt^2} + y = -x$$

given that at $t = 0$, $x = 2$, $y = -1$, $\dfrac{dx}{dt} = 0$, and $\dfrac{dy}{dt} = 0$ (15%)

3.

(3a) Consider a linear system **Ax** = **b**, where $\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 6 \\ -1 \end{bmatrix}$

Find the QR decomposition of the matrix **A** and the least squares solution of the above system. (5%)

(3b) Let the eigenvectors be v_1, v_2, and v_3, describing the general solution to the differential equation

$\dfrac{du}{dt} = \mathbf{K}u$, where $\mathbf{K} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$; Find the eigenvalues of the matrix **K.** (5%)

(3c) At what time T is the solution u (T) guaranteed to equal its initial value u (0)? (5%)

4. Given a vector function $\vec{E} = \vec{i}xy + \vec{j}(3x - y^2)$, and the coordinates of three points $P_1(5, 6)$, $P_2(5, 3)$ and $P_3(3, 3)$, where $\vec{i}$ and $\vec{j}$ are unit vectors along x- and y- axes respectively.

(4a) Evaluate the integral $\int \vec{E} \cdot d\vec{l}$ from P_1 straight to P_3. (5%)

(4b) Evaluate the integral $\int \vec{E} \cdot d\vec{l}$ from P_1 to P_3 along the piecewise straight path $P_1 P_2 P_3$, i.e. integrate from P_1 along straight-line segment to P_2 and then along another straight-line from P_2 to P_3. (5%)

(4c) Is this $\vec{E}$ a conservative field? And why? (5%)

5. Consider the following partial differential equation for the function $\varphi(\xi,t)$

$$\omega_0^{\ 2}\ \varphi_{\xi\xi} - \varphi_{tt} = 0$$

The boundary conditions are:

$$\omega_0^{\ 2}\ \varphi_{\xi}(0,t) - \varepsilon\varphi_{tt}(0,t) = 0$$

$$\omega_0^{\ 2}\ \varphi_{\xi}(1,t) + \varepsilon\varphi_{tt}(1,t) = 0$$

In the above equations, ω_0 and ε are constants.

(5a) Assume that $\varphi(\xi,t) = \Phi(\xi)T(t)$, find the time free behavior equation $\Phi(\xi)$ and the boundary conditions by the method of separating variables or the product method. (3%)

(5b) Write the eigenvalue equation of the system from the time free behavior equation and boundary conditions. (3%)

(5c) Find the complete eigenvalues, when $\varepsilon = 0$. (3%)

(5d) From the first nonzero eigenfunction $\Phi_1(\xi)$, find the behavior equation with time $T_1(t)$ and the corresponding solution $\varphi_1(\xi,t)$ (3%)

6. Consider the following ordinary differential equation

$$\ddot{y} + 2D\dot{y} = x(t)$$

where D is a constant. The right hand side is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ \exp(-2Dt) & t \geq 0 \end{cases}$$

Solve the equation by the method of Fourier transformation:

$$Y(i\omega) = F(i\omega) \cdot X(i\varpi)$$

(6a) Determine the complex input spectrum X(iω). (2 %)

(6b) Find the relationship in the complex form of the system F(iω) and X(iω) (3 %)

(6c) Find the complex output spectrum Y(iω) (3 %)

(6d) Calculate in the spectrum density S(ω)=| Y(iω)| and the value S(0) and S(∞). (2%)

(6e) Calculate the solution in the time domain y(t) for t>0 through the inverse Fourier transform. (3%)

Remark:

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{i\omega(a+i\omega)^2} d\omega = \begin{pmatrix} 0 & ,t<0 \\ \dfrac{2\pi \cdot \gamma(2,at)}{a^2 \cdot \Gamma(2)} & ,t>0 \end{pmatrix}$$

$$\text{with } \Gamma(2) = \int_0^{\infty} te^{-t}dt, \quad \gamma(2,at) = \int_0^{at} \bar{t}e^{-\bar{t}}d\bar{t},$$

$$\int xe^{-x}dx = \frac{-(x+1)}{e^x}$$

7 Find all Laurent series of

$$\frac{1}{z^2+2z}$$

with center $z = 0$, and indicate their respective regions of convergence. (9%)

8 Evaluate the following integral

$$\oint_C \mathrm{Im}(z)dz$$

where C is the unit circle around $z = 0$ in the complex plane. (6%)