科目：代　數(1004)　　校系所組：清大數學系純粹數學組

1. [10%] Let D_n denote the dihedral group D_n of order $2n$. Find the order of the center of D_n for any $n \geq 3$.

2. [10%] Let H, K be subgroups of a group G. Suppose that $Hx = Ky$ for some $x, y \in G$. Prove that $H = K$.

3. [10%] Let N, M be ideals of a ring R. The product NM of N and M is defined to be the subset of finite sums of the form

$$\sum_{i=1}^{n} x_i y_i$$

for $x_i \in N$, $y_i \in M$ and $n \in \mathbb{N}$. Prove that NM is also an ideal of R.

4. An element of a group G is called *torsion* if it is of finite order.
   (1) [10%] Suppose G is abelian. Prove that the subset of torsion elements is a subgroup of G.
   (2) [10%] Let $\mathrm{GL}_2(\mathbb{Z})$ denote the group of invertible 2×2 matrices over $\mathbb{Z}$. Prove that the torsion elements of $\mathrm{GL}_2(\mathbb{Z})$ do not form a subgroup.

5. Let R be an integral domain. Two elements $a, b \in R$ are called *associates in* R if $a = bu$ for some unit $u \in R$.
   (1) [10%] Prove that $-2+\sqrt{7}$ and $5-2\sqrt{7}$ are associates in the integral domain $\mathbb{Z}[\sqrt{7}]$ where
   $$\mathbb{Z}[\sqrt{7}] := \{\, m + n\sqrt{7} \mid m, n \in \mathbb{Z}\,\}.$$
   (2) [10%] Prove that an integral domain R is a field if and only if any two nonzero elements of R are associates in R.

6. An *automorphism* of a field F is a (ring) isomorphism $\sigma : F \to F$ of F with itself.
   (1) [10%] Prove that the field $\mathbb{Q}$ has no automorphism except the identity map.
   (2) [10%] Prove that the field $\mathbb{Q}(\alpha)$ where $\alpha^3 = 2$ has no automorphism except the identity map.
   (3) [10%] Find all automorphisms of $\mathbb{Z}_5(\alpha)$ where $\alpha^3 = 2$ and $\mathbb{Z}_5$ denotes the finite field of 5 elements.