科目：高等微積分(1001) 校系所組：中央大學數學系（甲組）
清華大學數學系（應用數學組、純粹數學組）

Advanced Calculus Written Exam, 2010

1. (10 points) Given the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as
$$f(x,y) = \frac{x^2y^3}{x^4+y^4} \text{ if } (x,y) \neq (0,0); \quad f(x,y) = 0 \text{ if } (x,y) = (0,0).$$
Is f differentiable at the point $(0,0)$? Give your reasons.

2. (15 points) Let A and B be nonempty closed subsets of $\mathbb{R}$. Consider subsets $A+B$, $A \times B$ of $\mathbb{R}$ defined by
$$A+B = \{a+b : a \in A, b \in B\}, \quad A \times B = \{ab : a \in A, b \in B\}.$$
Prove or disprove (by providing a counterexample) that $A+B$, $A\times B$ are closed.

3. (12 points) Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^6 = 1\}$. Is S compact? Is it connected? You must justify your answers by quoting appropriate theorems.

4. (13 points) Let A and B be two $n \times n$ real matrices. Define $f : \mathbb{R}^n \to \mathbb{R}$ by
$$f(x) = \langle Ax, Bx \rangle, \quad x \in \mathbb{R}^n.$$
Use the **definition of derivative** to calculate the derivatives $Df(x)$ and $D^2f(x)$. The answers should be expressed in terms of A, B, and x.

5. (12 points) Assume that $f(x)$ is continuous on $[0,\infty)$ with $f(x) \to 1$ as $x \to \infty$ (note that in general $f(x)$ may not be differentiable). Evaluate the following limit
$$\lim_{t\to 0^+} t \int_0^\infty e^{-tx} f(x)\,dx$$
and give your reasons.

6. (10 points) Given the sequence of functions
$$f_n(x) = \frac{2nx}{1+n^2x^2}, \quad x \in [0,1], \quad n = 1, 2, 3, \ldots,$$
does it converge uniformly on $[0,1]$ as $n \to \infty$? Give your reasons.

7. (13 points) Show that (by quoting appropriate theorem) from the equations
$$x^2 - y^2 - u^3 + v^2 + 4 = 0, \qquad 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$
we can solve u, v as continuously differentiable functions of (x,y) for (x,y) near $(2,-1)$ satisfying $u(2,-1) = 2$, $v(2,-1) = 1$. Also find the value of $\frac{\partial u}{\partial x}(2,-1) + \frac{\partial v}{\partial x}(2,-1)$.

8. (15 points) Assume $f \in C^2(a,\infty)$, $a \in \mathbb{R}$.

   (a) (10 points) Let M_0, M_1 and M_2 be the supremum of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ on (a,∞). Prove the following "**interpolation formula**":
   $$M_1^2 \le 4M_0M_2.$$
   Hint: Use Taylor's formula: for any $h > 0$ and $x \in (a,\infty)$, we have
   $$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(\xi)h^2, \quad \xi \in (x, x+h).$$

   (b) (5 points) Assume $f \in C^2(a,\infty)$ and satisfies $f(x) \to C$ (C is some constant) as $x \to \infty$ and $|f''(x)| \le M$ for all $x \in (a,\infty)$. Prove that $f'(x) \to 0$ as $x \to \infty$.