科目：代數(1004)

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ALGEBRA

In the problems below, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 1\}$, $\mathbb{Z}_m$ denotes the additive group of integers modulo a positive integer m, $\mathbb{Q}$ (resp. $\mathbb{C}$) denotes the field of rational numbers (resp. complex numbers).

(17%) 1. Let $< (3,3) >$ be the cyclic subgroup of $\mathbb{Z}_9 \times \mathbb{Z}_6$ generated by $(3,3)$ and let L be the quotient group $\mathbb{Z}_9 \times \mathbb{Z}_6 \ / < (3,3) >$.

(9%) (a) Find the order $|L|$.

(8%) (b) Prove that L is not a cyclic group.

(20%) 2. Let G be a finite group with $|G| = p^n$ $(n \geq 2)$, where p is a prime, and let $Z(G)$ be its center.

(10%) (a) Prove that $|Z(G)| = p^k$ for some $k \geq 1$.

(10%) (b) Prove that there exists a normal subgroup H of G with $|H| = p$.

(23%) 3. Let $D = \{a+b\sqrt{3}\,i \mid a, b \in \mathbb{Z} \text{ or } a = \frac{x}{2}, b = \frac{y}{2} \text{ with both } x \text{ and } y \text{ odd integers}\}$. Here $\sqrt{3}\,i = \sqrt{-3}$. It is known that D is a subring of $\mathbb{C}$ and is a Euclidean domain relative to the norm $N(a+b\sqrt{3}\,i) = a^2 + 3b^2 \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$.

(3%) (a) Show that $a^2 + 3b^2 = 1$ implies $a + b\sqrt{3}\,i$ is a unit in D.

(8%) (b) Prove that $1 + \sqrt{3}\,i$ is an irreducible element of D.

(12%) (c) Let $< 1 + \sqrt{3}\,i >$ be the ideal of D generated by $1 + \sqrt{3}\,i$. Prove that there are exactly 4 elements in the quotient ring $D \ / < 1 + \sqrt{3}\,i >$.

(18%) 4. Let R be a commutative ring with identity.

(11%) (a) Let S be a non-empty multiplicative subset of R (means $x, y \in S \Rightarrow xy \in S$) such that $0 \notin S$. By Zorn's lemma, $\exists$ an ideal P such that (i) $P \cap S = \emptyset$ and (ii) for any ideal $Q \subset R$, $Q \underset{\neq}{\supset} P \Rightarrow Q \cap S \neq \emptyset$. Prove that P is a prime ideal of R.

(7%) (b) Prove that if $x \in R$ is an element that lies in every prime ideal of R then x is nilpotent (means $x^n = 0$ for some $n \in \mathbb{N}$).

(22%) 5. For a prime p, let $\zeta = \cos\frac{2\pi}{p} + i\ \sin\frac{2\pi}{p} \in \mathbb{C}$ (so $\zeta \neq 1, \zeta^p = 1$) and let $K = \mathbb{Q}(\zeta)$ be the extension field of $\mathbb{Q}$ generated by ζ.

(5%) (a) Let $u \in \mathbb{C}$ be an element such that $u^p \in K$ and also $u^m \in K$ for some $m \in \mathbb{N}$ with $1 \leq m < p$. Prove that $u \in K$.

(5%) (b) Show that if $x^p - a \in K[x]$ has a root in K then it splits over K.

(12%) (c) For any $a \in K$, prove that the polynomial $x^p - a \in K[x]$ is either irreducible or splits over K.