國立清華大學100學年度碩士班入學考試試題

系所班組別: 數學系碩士班應用數學組
考試科目 (代碼): 線性代數 (0202)

 請在 [答案卷、卡]作答

1. [20%] True or false? With a reason.

(1) The following two matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are similar.

(2) Let A, B be two m by n matrices over $\mathbb{R}$. Then

$$\mathrm{Col}(A + B) = \mathrm{Col}(A) + \mathrm{Col}(B)$$

where $\mathrm{Col}(X)$ denote the column space of a matrix X.

2. [10%] Let V denote the space of 3 by 3 matrices A over $\mathbb{R}$ such that $AS = SA$ where

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find the dimension of V.

3. [10%] Let V denote the complex vector space of 2 by 2 matrices over $\mathbb{C}$. Let $T\colon V \to V$ be the linear transformation given by $T(X) = AX$ where

$$A = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}.$$

Find the rank of T.

4. [10%] Find the minimal polynomial of the 6 by 6 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

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5. [10%] Find the singular value decomposition of the 2 by 3 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

6. [20%] Let T be a linear operator on $\mathbb{R}^n$ and let v be a nonzero vector in $\mathbb{R}^n$. The polynomial $f(x)$ is called a *T-annihilator for* v if $f(x)$ is a monic polynomial of least degree for which $f(T)(\mathrm{v}) = \vec{0}$.

(1) Prove that the T-annihilator for v is unique.

(2) Find the T-annihilator for $\mathrm{v} = (1, \sqrt{2}, 1) \in \mathbb{R}^3$ where

$$T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

7. [20%] Let V be a finite-dimensional complex inner product space and U is a unitary operator on V such that $U(\mathrm{v}) = \mathrm{v}$ implies $\mathrm{v} = \vec{0}$.

(1) Show that $I - U$ is invertible.

(2) Show that $(I + U)(I - U)^{-1} = (I - U)^{-1}(I + U)$.

(3) Show that the linear operator $\sqrt{-1}(I + U)(I - U)^{-1}$ is self-adjoint.

(4) For a self-adjoint operator T on V, show that $(T - \sqrt{-1}I)(T + \sqrt{-1}I)^{-1}$ is unitary.