國立清華大學100學年度碩士班入學考試試題

系所班組別: 數學系碩士班純粹數學組
考試科目 (代碼): 代數與線性代數 (0102)

 請在 [答案卷、卡]作答

1. [20%] True or false? With a reason.

(1) Let G be a group of order 40. If $g \in G$ such that $g^{20} \neq e$ (where e is the identity of G), then G is cyclic and generated by g.

(2) Let A, B be two m by n matrices over $\mathbb{R}$. Then

$$\mathrm{Col}(A+B) = \mathrm{Col}(A) + \mathrm{Col}(B)$$

where $\mathrm{Col}(X)$ denote the column space of a matrix X.

2. [10%] Let $C_1, \ldots, C_k$ be the conjugacy classes of a finite group. Show that each product C_iC_j is a union of conjugacy classes.

3. [10%] Let $f(x), g(x) \in \mathbb{Q}[x]$ (the polynomial ring over $\mathbb{Q}$), and let $d(x)$ be the greatest common divisor of $f(x)$ and $g(x)$. Show that

$$\langle f(x)\rangle + \langle g(x)\rangle = \langle d(x)\rangle$$

where $\langle h(x)\rangle$ denotes the ideal generated by the polynomial $h(x)$.

4. [10%] Let $\mathbb{H}$ denote the quaternions $\{\, a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\,\}$. There exists a matrix $K \in M_2(\mathbb{C})$ such that $\phi\colon \mathbb{H} \to M_2(\mathbb{C})$ defined by

$$\phi(a+bi+cj+dk) = a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} + c\begin{bmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{bmatrix} + dK$$

for $a,b,c,d \in \mathbb{R}$, gives an isomorphism of $\mathbb{H}$ with $\phi[\mathbb{H}]$.

(1) Find the matrix K.

(2) Find another homomorphism $\psi\colon \mathbb{H} \to M_2(\mathbb{C})$ such that ψ gives an isomorphism $\mathbb{H}$ with $\psi[\mathbb{H}]$.

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5. [10%] Let E be a finite extension of a field F. Let $\alpha \in E$ be algebraic of odd degree over F. Show that $F(\alpha) = F(\alpha^2)$.

6. [20%] Let T be a linear operator on $\mathbb{R}^n$ and let v be a nonzero vector in $\mathbb{R}^n$. The polynomial $f(x)$ is called a T-*annihilator for* v if $f(x)$ is a monic polynomial of least degree for which $f(T)(\mathrm{v}) = \vec{0}$.

(1) Prove that the T-annihilator for v is unique.

(2) Find the T-annihilator for $\mathrm{v} = (1, \sqrt{2}, 1) \in \mathbb{R}^3$ where

$$T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

7. [20%] Let V be a finite-dimensional complex inner product space and U is a unitary operator on V such that $U(\mathrm{v}) = \mathrm{v}$ implies $\mathrm{v} = \vec{0}$.

(1) Show that $I - U$ is invertible.

(2) Show that $(I + U)(I - U)^{-1} = (I - U)^{-1}(I + U)$.

(3) Show that the linear operator $\sqrt{-1}(I + U)(I - U)^{-1}$ is self-adjoint.

(4) For a self-adjoint operator T on V, show that $(T - \sqrt{-1}I)(T + \sqrt{-1}I)^{-1}$ is unitary.