科目：應用數學(3001)　校系所組：中大照明與顯示科技研究所甲組、天文研究所
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1 (1a) Solve the differential equation $\frac{dy}{dt} = e^{y+t}$ for $y(t)$. (11 points)

(1b) Find the general solution of $\frac{d^2y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + \frac{6}{x^2}y = \frac{4}{x^2}$. (11 points)

(1c) A tank initially contains 40 g of salt mixed in 100 liters of water. A solution contains 4 g of salt per liter is pumped into the tank at a rate of 5 liter/min. The stirred mixture flows out the tank at the same rate. How much salt is in the tank after 20 minutes? (12 points)

2 (17 points)

(a) Given a 2×2 hermitian matrix $B = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$, and $x = \begin{pmatrix} a \\ b \end{pmatrix}$ as a column matrix, (i) solve the eigenvalue problem $Bx = \lambda x$ for the eigenvalues and eigenvectors, and then (ii) find a matrix S such that $B_s = SBS^{-1}$ becomes a diagonized matrix.

(b) Prove that $\det \exp[iA] = \exp[iTrA]$ for any $n \times n$ hermitian matrix A. Prove this result from the fact that any hermitian matrix A can always be diagonized.

[Remark: $\det B$ and $Tr\,B$ are the determinant and the trace of the matrix B respectively. θ, a and b are constant parameters. $\exp[A] = \sum_{k=0}^{\infty}[A]^k/k!$ defines the exponential mapping of any matrix A.]

3 (16 points)

Define a $27(= 3 \times 3 \times 3)$ components 3-index function A_{ijk} for all $i, j, k = 1, 2, 3$ as a totally antisymmetric 3-index function with $A_{123} = 1$. Here totally antisymmetric means that $A_{ijk} = -A_{jik} = -A_{ikj}$ for all $i, j, k = 1, 2, 3$.

(a) (i) List all non-vanishing components of A_{ijk}, e.g. $A_{132} = -1$, and $\cdots$ from the antisymmetric properties of A_{ijk}. (ii) Also give a reason, from the symmetric properties of A_{ijk}, to explain why there are totally $k(=?)$ non-vanishing components.

(b) The determinant of any 3×3 matrix B can be defined as $\det B = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk}\,[B_{1i}B_{2j}B_{3k}]$. From the symmetric properties of A_{ijk}, show that $A_{lmn}\det B = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} A_{ijk}\,[B_{li}B_{mj}B_{nk}]$.

[Remark: This definition agrees with the conventional definition of the determinant of any 3×3 matrix B that $\det B = \det\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = B_{11}[B_{22}B_{33} - B_{23}B_{32}] - B_{21}[\cdots] + B_{31}[\cdots].$]

4 Consider the function

$$f(z) = \frac{1}{1+z}.$$

(a) Expand $f(z)$ about the point $z = 0$. What is the convergence radius? (5 points)

(b) Expand $f(z)$ about the point $z = i$. What is the convergence radius? (5 points)

5 Perform the following integrals.

(a) For $\omega > 0$ and t is real, calculate

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-ikt}}{k^2+\omega^2}dk. \quad \text{(8 points)}$$

(b) For $\sigma > 0$, calculate

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x\ \sin x}{x^2-\sigma^2}dx. \quad \text{(10 points)}$$

(c) For $\sigma > 0$ and $\epsilon > 0$, calculate

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x\ \sin x}{x^2-(\sigma-i\epsilon)^2}dx. \quad \text{(5 points)}$$