1. (15%) For the following three questions, please find the true statements. (Proofs are not needed and no partial credits will be given for each question.)

   (I) (5%) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are elements of a vector space V and W is a subset of V.

   (A). W forms a subspace of V.

   (B). If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly dependent, then each $\boldsymbol{v}_i$, where $1 \le i \le n$, can be expressed as a linear combination of the rest $(n-1)$ vectors.

   (C). If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span V, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a minimal spanning set if and only if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent.

   (D). If $a\boldsymbol{v}_1 = b\boldsymbol{v}_1$, then $a = b$, where a and b are both scalars.

   (E). If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis of V, and W is a subspace of V, we may find a set of basis vectors of W from $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$.

   (II) (5%) Let A and B be two $n \times n$ matrices and $\boldsymbol{x}$ be an $n \times 1$ column vector.

   (A). If A and B are both diagonalizable, then A and B commute.

   (B). If A is diagonalizable, then A has at least one eigenvalue.

   (C). If λ is an eigenvalue of A, $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has only trivial solutions.

   (D). If A is symmetric, it has real eigenvalues and is diagonalizable.

   (E). If A and B are both nonsingular, there exists a unique inverse matrix of AB.

   (III) (5%) Let L_1 and L_2 be linear transformations from R^2 into R^2, where R is the set of real numbers.

   (A). If $L_1(\boldsymbol{x}_1) = L_1(\boldsymbol{x}_2)$, then vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must be equal.

   (B). If $\boldsymbol{x} \in \ker(L_1)$, where $\ker(L_1)$ is the kernel of L_1, then $L_1(\boldsymbol{x} + \boldsymbol{v}) = L_1(\boldsymbol{v})$ for all $\boldsymbol{v} \in R^2$.

   (C). If $L_1 + L_2$ is the mapping described by $(L_1 + L_2)(\boldsymbol{v}) = L_1(\boldsymbol{v}) + L_2(\boldsymbol{v})$ for all $\boldsymbol{v} \in R^2$, then $L_1 + L_2$ is also a linear transformation.

   (D). If L_1 rotates each vector by 60° and then reflects the resulting vector about the x-axis and L_2 also does the same two operations but in the reverse order, then $L_1 = L_2$.

   (E). Let A be the standard matrix representation of L_1. If L_1^2 is defined by $L_1^2(\boldsymbol{x}) = L_1(L_1(\boldsymbol{x}))$ for all $\boldsymbol{x} \in R^2$, then L_1^2 is a linear transformation and its standard matrix representation is A^2.

2. (20%) Consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

(a) (4%) Find the determinant of A.

(b) (4%) Find the rank of A.

(c) (4%) Determine a basis for the column space of A^T.

(d) (4%) Determine a basis for the nullspace of A.

(e) (4%) Find eigenvalues of A.

3. (15%) If an $n \times n$ matrix A has fewer than n linearly independent eigenvectors, we say that A is *defective*. For each of the following matrices, find all possible values of the scalar α that make the matrix defective or show that no such values exist.

(a) (5%) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & \alpha \end{bmatrix}$

(b) (5%) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{bmatrix}$

(c) (5%) $\begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & \alpha \end{bmatrix}$

4. (30%) The joint probability density function of random variables of X and Y is given by

$$f(x,y) = \frac{1}{2\pi} \exp[-(x^2 - \sqrt{3}xy + y^2)], \qquad -\infty < x < \infty,\ -\infty < y < \infty.$$

(a) (5%) Find the marginal probability density function of X.

(b) (5%) Find the conditional mean of Y, given that $X = x$.

(c) (5%) Find the conditional variance of Y, given that $X = x$.

(d) (5%) Find the joint moment-generating function of of X and Y given by

$$M_{X,Y}(t_1, t_2) = E[\exp(t_1 X + t_2 Y)].$$

(e) (10%) Show that $X + Y$ and $X - Y$ are independent random variables.

5. (10%) Let $X_1, X_2, \ldots$ be a sequence of independent and exponentially distributed random variables with mean 1. Find $E[N]$ when

$$N = \max\left\{n : \sum_{i=1}^{n} X_i \leq 1\right\}.$$

(If $\{n : \sum_{i=1}^{n} X_i \leq 1\}$ is an empty set, then $N = 0$.) Note that the probability density function of an exponential random variable X with parameter λ is given by

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

6. (10%) Let $X_0, X_1, X_2, \ldots$ be a sequence of independent Poisson random variables with mean 1. Let $Y_n = X_0 + X_n$ for all n and $S_n = \sum_{i=1}^{n} Y_i$. Find

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - 1\right| < \frac{1}{4}\right).$$

Note that the probability mass function of a Poisson random variable X with parameter λ is given by

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots.$$