

科目：應用數學(3001) 校系所組：中大物理學系、天文研究所

交大電子物理學系丙組、物理研究所

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Applied Mathematics

Show your calculation steps clearly.

Problem 1 (20 points)

(1a) Write down the definition of a group. (~ 4 points)

(1b) Let U be a group element of the unitary group $U(3, C)$ (also written as $U(3)$) and define G by

$$U \equiv \exp[G] \equiv \sum_{n=0}^{\infty} \frac{G^n}{n!}$$

write down a set of complete basis $\{T_i\}$ that spans the space of Lie algebra $u(3, C)$. [Hint: $G = \sum_{i=1}^N G_i T_i, \forall G \in u(3, C)$ and $G_i \in \mathbf{R}$. You must answer $N = ?$ and list the properties of G_i .] (~ 8 points)

(1c) For a set of nonabelian operators A, B , and C , $[A, \{B, C\}] = [\{A, B\}, C] - \{B, \{A, C\}]$ is known as the Jacobi identity. Show that there is another expression for the Jacobi identity: $[A, \{B, C\}] = \{[A, B], C\} + Y$ by expressing Y as similar linear combination of commutators ($[,]$) and/or anticommutators ($\{, \}$) of A, B , and C . [Hint: Y must be derived or be provided with a geometric (or statistical) interpretation, not just writing it down out of your memory.] (~ 8 points)

Problem 2 (15 points)

Show that

$$\pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

by integrating $\int_0^2 f(x) dx$ with a corresponding function $f(x)$. [Hint: $f(x)$ must be derived from converting the series into an integral. 7 points if you derive a compact expression for $f(x)$.]

Problem 3 (10 points)

Let $\Omega = \frac{N!}{n_1! n_2! \dots n_K!} g_1^{n_1} \dots g_K^{n_K}$ where g_i 's are constants. Here $N, n_i \gg 1$. Assume there exist two constraints $\sum n_i = N$ and $\sum \epsilon_i n_i = E$ where N and E are two constants. Show that $n_i = g_i C e^{-\beta \epsilon_i}$ when Ω is at a maximum (C and β are constants). (Hint: Use the equivalent condition that $\ln \Omega$ is at a maximum. In $N! \approx N \ln N - N$ for large N .)

注意：背面有試題

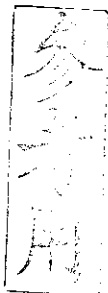
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Problem 4 (10 points)

(4a) For a contour map $h = 0.001x^2 + 0.002y^2$, find the change of height δh for the displacement $\delta \mathbf{r} = (2, 1)$ at $\mathbf{r} = (x=100, y=300)$. (4 points)

(4b) Along what direction is the largest slope at this position? (3 points)

(4c) Show that ∇h is always perpendicular to the constant h contour. (3 points)

Problem 5 (10 points)

(5a) For $N = ak^2$ and $\omega = \sin(bk)$ where a and b are constants, find $\frac{dN}{d\omega}$.

(5b) If $f(x) = 3x + 5$ and $g(x^2 - 1) = f(x - 7)$, find the value of $g(6)$. (5 points each)

Problem 6 (35 points)

(6a) Discuss the analytic properties of the following functions in the complex z -plane and, if they have singularities, state the nature of the singularity (branch points, poles, etc) (12 points)

(i) e^z ;

(ii) $\frac{1}{\sqrt{z-i}}$;

(iii) z^* .

(6b) Use Cauchy's theorem to evaluate the following contour integral where C is the circle (i) $|z| = 2$; (ii) $|z| = \frac{1}{2}$. (12 points)

$$\oint_C \frac{e^z}{(z-1)^2} dz.$$

(6c) Use the method of Fourier transform to solve

$$\frac{d^2 y}{dx^2} - y = -\theta(1 - |x|)$$

for $-\infty < x < \infty$, with $y(x) \rightarrow 0$ and $\frac{dy}{dx} \rightarrow 0$ as $|x| \rightarrow \infty$. Note that

$$\theta(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases}$$

is the Heaviside function. (11 points)