

國立清華大學命題紙

99 學年度工程與系統科學系乙組、核子工程與科學研究所甲組、聯合招生(工科丙組、先進光源工
科組)碩士班入學考試 *請在【答案卷卡】作答

科目 工程數學 科目代碼 2701、2801、9801 共 2 頁，第 1 頁

1. Solve the ordinary differential equation $x^2y'' + 4xy' + (x^2 + 2)y = 0$, using Frobenius method.
(10%)

2. Let's denote $J_\nu(x)$ and $Y_\nu(x)$ to be the Bessel functions of 1st kind and of 2nd kind, respectively. I recall you that $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ is called Bessel's equation where ν is a real and nonnegative number. Find a general solution for the ordinary differential equation $xy'' + 11y' + xy = 0$ in terms of $J_\nu(x)$ and $Y_\nu(x)$.

[Hint: use the substitution $y = x^{-5}u$ in your derivation.]

(5%)

3. The Legendre polynomials of degree smaller than 6 are given as follow.

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Develop $(x + 1)^2$ in Fourier-Legendre series.

(5%)

4. Fourier transform of $f(x)$ is defined as $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$. Find the Fourier transform of

$$f(x) = e^{-ax^2} \quad (a > 0).$$

(5%)

5. (a) Calculate the flux of vector

$$\mathbf{F} = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$$

out of the unit sphere.

(12%)

(b) Find a unit vector normal to the surface S given by

$$z = x^2y^2 + y + 2$$

at the point (1,0,1).

(13%)

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6. The temperature in a rod of unit length in which there is heat transfer from both ends, $x = 0$ and $x = 1$, into a surrounding medium kept at a constant temperature 0 is determined from

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0, \quad k > 0$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = hu(0, t), \quad h > 0, \quad t > 0$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=1} = -hu(1, t), \quad h > 0, \quad t > 0$$

The initial temperature in the rod is $f(x)$ throughout. Solve for $u(x, t)$.
(13%)

7. (a) Show that the value of an analytic function at any point z_0 is the average of its values on any circle, with z_0 as its center, which lies inside the region of analyticity.

(6%)

- (b) Find all possible values of $(1+i)^i$.

(6%)

8. Laplace transform of $f(t)$ is defined as $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$. Find the solution of the following equation using Laplace Transform.

$$y'' + 2y' + 2y(t) = \delta(t-2), \quad y(0) = 0, \quad y'(0) = 1$$

where $\delta(t-2)$ is the unit impulse function

$$\delta(t-2) = \begin{cases} \infty & \text{if } t=2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^{\infty} \delta(t-2) dt = 1,$$

$$\text{and } L\{\delta(t-a)\} = e^{-as}.$$

(12%)

9. Obtain the general solution $x(t)$, $y(t)$ for the following coupled differential equations

$$x' = x(t) - 3y(t).$$

$$y' = -x(t) - y(t) - 4t^2$$

(13%)